Math 725 | QUICK REVIEW SO FAR OF: Chapter 3 (Brézis's book).

3.1 The coarsest topology for which a collection of maps becomes continuous

general

$(\varphi_i)_{i\in I}$ $\quad \varphi_i : X \to Y_i$ $\quad$ X set without any structure

$(Y_i)_{i\in I}$ collection of topological spaces.

- We constructed a topology $\mathcal{Z}$ for X which was the most economical (fewest open sets)
  → coarsest/weakest topology associated to $(\varphi_i)_{i\in I}$

- Prop 3.1: $(x_n) \subset X$, $x_n \to x$ in $\mathcal{Z}$ $\iff$

  $\varphi_i(x_n) \to \varphi_i(x)$ $\forall i \in I$

- Prop 3.2: Let Z be a topological space and $\psi : Z \to X$. Then ψ is continuous $\iff$ $\varphi_i \circ \psi : Z \to Y_i$ is continuous $\forall i \in I$.

3.2 Definition and Properties of $\sigma(E, E^*)$ the weak TOPOLOGY

E = Banach space $\quad E^*$ = dual of E (Banach)

$(E, \|\cdot\|)$ STRONG OR NORM TOPOLOGY

Notation: for $f \in E^*$, $\varphi_f : E \to \mathbb{R}$ is the linear functional $\varphi_f(x) := \langle f, x \rangle$.

$(\varphi_f)_{f \in E^*}$ is a collection of functionals $: E \to \mathbb{R}$.

Definition: The weak topology $\sigma(E, E^*)$ on E is the coarsest topology associated to the collection $(\varphi_f)_{f \in E^*}$ in the sense of the previous section 3.1. with $E = X$, $Y_i = \mathbb{R}$ $\forall i \in I = E^*$

Remark: The weak topology is weaker than the norm topology: note that every map φ_f is continuous in the usual topology (norm).

Prop. 3.3: The weak topology is Hausdorff.

(Recall a space V is said to be Hausdorff if $\forall\, v, w \in V$, $v \neq w$ $\exists$ disjoint open sets $\mathcal{U}_v$ and $\mathcal{U}_w$ $v \in \mathcal{U}_v$ $w \in \mathcal{U}_w$)

Prop 3.4: Let $x_0 \in E$. Given $\varepsilon > 0$ and a FINITE set $\{f_1, f_2, \dots, f_k\}$ in E^* consider

$$V = V(f_1, f_2 \dots, f_k ; \varepsilon) =$$

$$\{x \in E : |\langle f_i, x - x_0 \rangle| < \varepsilon \ \forall i = 1, \dots k\}$$

Then V is a neighborhood of x_0 for the $\sigma(E, E^*)$ topology. Moreover we obtain a <u>BASIS OF NEIGHBORHOODS</u> of x_0 for $\sigma(E, E^*)$ by varying ε, k and the $f_i \in E^*$.

<u>PROP. 3.5</u> Let $(x_n) \subset E$ sequence. Then.

(i) $x_n \rightharpoonup x \iff \langle f, x_n \rangle \to \langle f, x \rangle \ \forall f \in E^*$

$x_n \rightharpoonup x$ means x_n converges weakly to x in $\sigma(E,E^*)$ topology.

(ii) $x_n \to x$ (ie strongly or norm topology) $\Rightarrow$ $x_n \rightharpoonup x$

(iii) $x_n \rightharpoonup x \Rightarrow (\|x_n\|)$ is bounded and $\|x\| \leq \liminf \|x_n\|$.

(iv) $x_n \rightharpoonup x$ in $\sigma(E, E^*)$ and $f_n \to f$ strongly in E^* then

$$\langle f_n, x_n \rangle \longrightarrow \langle f, x \rangle$$

<u>PROP 3.6</u>: When E is <u>FINITE</u> dimensional strong $\iff$ weak

ie. $x_n \rightharpoonup x \iff x_n \to x$.

Remark 2: Open (closed) sets in the weak topology $\sigma(E,E^*)$ are always open (closed) in the strong topology. In any infinite-dimensional space the weak topology is STRICTLY COARSER than the strong topology; ie. $\exists$ open (closed) sets in the strong topology that are NOT open (closed) in the weak topology. Examples:

Ex 1) The unit sphere $S = \{x \in E : \|x\| = 1\}$ with E infinite-dimensional is NEVER closed in the weak topology $\sigma(E,E^*)$. More precisely, we have

(1) $$\overline{S}^{\sigma(E,E^*)} = B_E \qquad \text{where}$$
(closure of S in $\sigma(E,E^*)$)

$$B_E = \text{closed unit ball in } E = \{x \in E, \|x\| \leq 1\}$$

Remark: In infinite-dimensional spaces, the weak topology is never metrizable: ie there is no metric (and hence no norm) on E that induces on E the weak topology $\sigma(E,E^*)$. But if E^* is separable one can define a norm on E that induces on BOUNDED SETS of E the weak topology $\sigma(E,E^*)$.

Example 2: The unit ball $\mathcal{U} = \{ x \in E ; \|x\| < 1 \}$ with E infinite-dimensional is NEVER OPEN in the weak topology $\sigma(E, E^*)$. Suppose, by contradiction $\mathcal{U}$ is weakly open. Then its complement $\mathcal{U}^c =$
$= \{ x \in E : \|x\| \geq 1 \}$ is weakly closed. It follows that $S = B_E \cap \mathcal{U}^c$ is also weakly closed; contradicting Example 1.

Remark: In infinite dimensions, $\exists$ sequences converging weakly that do not converge strongly (if E^* separable or E is reflexive. can construct $(x_n) \subset E$, $\|x_n\| = 1$ $x_n \rightharpoonup 0$) Note ℓ^1 is unusual b/c $\Rightarrow$ is true but this case is rare.)

$\longrightarrow$ $\boxed{3.3}$: Weak Topology + Convex Sets

unless : convex sets.

Weakly open/closed $\Longrightarrow$ strongly open/closed

$\not\Longleftarrow$ (infinite dimensions)

Unless the set is also convex.

Theorem 3.7 : Let C be a convex subset of E. Then C is weakly closed $\Longleftrightarrow$ is strongly closed.

Corollary 3.8 (MAZUR'S): Assume $x_n \rightharpoonup x$. Then $\exists$ a sequence (y_n) made up of convex combinations of the (x_n) that converges strongly to x.

Corollary 3.9: Assume that $\varphi : E \to (-\infty, +\infty]$ is convex and l.s.c. in the strong topology. Then φ is l.s.c. in $\sigma(E, E^*)$ top. (weak).

In particular: φ convex and strongly continuous $\Rightarrow$ φ is weakly l.s.c. (Example: $\varphi(x) = \|x\|$. $x_n \rightharpoonup x \Rightarrow \|x\| \leq \liminf \|x_n\|$)

Read Theorem 3.10 & its proof

$\boxed{\to 3.4}$: The Weak* Topology $\sigma(E^*, E)$.

Def: The weak* topology $\sigma(E^*, E)$ is the coarsest topology on E^* associated to the collection $(\varphi_x)_{x \in E}$ with $X = E^*$ $Y_i = \mathbb{R}$ $(i \in I)$ $I = E$)

Remark. Since $E \subset E^{**}$, $\sigma(E^*, E)$ has fewer open sets (closed sets) than $\sigma(E^*, E^{**})$ which in turn has fewer open sets (resp closed sets) than the strong topology (norm top.)

(ie. is coarser)

Point: A coarser topology has more compact sets. For example the closed unit ball B_{E^*} in E^* which is NEVER COMPACT in the strong topology (unless $\dim E < \infty$) is always COMPACT in the weak* topology.

Prop 3.11: The weak* topology is Hausdorff.

Prop 3.12: Let $f_0 \in E^*$. Given a FINITE set $\{x_1, x_2, \dots, x_k\}$ in E and $\varepsilon > 0$, consider

$$V = V(x_1, \dots, x_k; \varepsilon) = \{f \in E^* ; |\langle f - f_0, x_i \rangle| < \varepsilon \quad \forall i = 1, \dots, k\}.$$

Then V is a nbhd of f_0 for the topology $\sigma(E^*, E)$. Moreover we obtain a basis of nbhds of f_0 for $\sigma(E^*, E)$ by varying ε, k, and $x_i \in E$.

NOTATION: $(f_n) \in E^*$ converges to f in the weak* topology ($\sigma(E^*, E)$) is denoted by $f_n \xrightarrow{*} f$.

Prop 3.13: Let (f_n) be a sequence in E^*. Then

(i) $f_n \xrightarrow{*} f$ in $\sigma(E^*, E) \iff \langle f_n, x \rangle \to \langle f, x \rangle \quad \forall x \in E$

(ii) If $f_n \to f \Rightarrow f_n \to f$ in $\sigma(E^*, E^{**})$

AND If $f_n \to f$ in $\sigma(E^*, E^{**}) \Rightarrow f_n \xrightarrow{*} f$ in $\sigma(E^*, E)$.

(iii) If $f_n \xrightarrow{*} f$ in $\sigma(E^*, E) \Rightarrow$

$(\|f_n\|_{E^*})$ is bounded and $\|f\|_{E^*} \leq \liminf \|f_n\|_{E^*}$

(iv) If $f_n \xrightarrow{*} f$ in $\sigma(E^*, E)$ and if $x_n \to x$ in E

then $\langle f_n, x_n \rangle \xrightarrow[n\to\infty]{} \langle f, x \rangle$

IMPORTANT — Read proof

PROPOSITION 3.14: Let $\varphi : E^* \to \mathbb{R}$ be a linear functional that is continuous for the weak* topology. Then $\exists$ some $x_0 \in E$ /

$$\langle f, x_0 \rangle = \varphi(f) \quad \forall f \in E^*.$$

MOST IMPORTANT FACT OF weak* top.

THEOREM 3.16 (BANACH - ALAOGLU - BOURBAKI)

The closed unit ball

$$B_{E^*} = \{ f \in E^* : \|f\|_{E^*} \leq 1 \}$$

is COMPACT in the weak* topology $\sigma(E^*, E)$.

To prove this theorem we need Tychonoff's theorem: Let $(A_\alpha)_{\alpha \in I}$ be a collection of compact spaces. Then $\prod_{\alpha \in I} A_\alpha$ is compact in the product (weak) topology.

Proof of [BA]B :

- Consider (Cartesian product) $Y := \mathbb{R}^E$ which consists of all maps : $E \longrightarrow \mathbb{R}$. We can denote the elements of Y by $w = (w_x)_{x \in E}$
- Equip Y with STANDARD product topology (M623/624). which is the coarsest topology in Y associated to the collection of maps $w \longmapsto w_x$ as $(x \in E)$.

→ [This is the same topology as pointwise convergence]

- Next consider E^* with the weak* topology $\sigma(E^*, E)$

→ [Recall E^* consists of all continuous linear maps from E to $\mathbb{R}$. $\Rightarrow E^* \subseteq Y$]

Let $\Phi : E^* \to Y$ be the cannonical injection from $E^* \longrightarrow Y$ / $\Phi(f) = (w_x)_{x \in E}$ and $w_x = \langle f, x \rangle$ $\left[\Phi(f) = (\langle f, x \rangle)_{x \in E}\right]$

- Φ is continuous by PROP 3.2 and the fact that $\forall x \in E$ fixed $f \longmapsto (\Phi(f))_x = \langle f, x \rangle$ is continuous.

- The inverse Φ^{-1} is also continuous from $\Phi(E^*)$ (equipped with the Y topology) into E^* again by PROP 3.2 since $\forall x \in E$, fixed, the map $\omega \longmapsto \langle \Phi^{-1}(\omega), x \rangle$ is continuous on $\Phi(E^*)$ [$\langle \Phi^{-1}(\omega), x\rangle = \omega_x$ since $\omega = \Phi(f)$ for some $f \in E^*$, so $\langle \Phi^{-1}(\omega), x \rangle = \langle f, x \rangle = \omega_x$

- All in all Φ is a homeomorphism $E^* \longrightarrow \Phi(E^*)$

- Define $K := \{ \omega \in Y \,/\, |\omega_x| \leq \|x\|,\ \omega_x + \omega_y = \omega_{x+y},\ \omega_{\lambda x} = \lambda \omega_x\ \forall \lambda \in \mathbb{R}\ \forall x, y \in E \}$

HMWK — Then $K = \Phi(B_{E^*})$ [WHY ?].

- To conclude then we need to show that K is compact. Write $K = K_1 \cap K_2$ where $K_1 := \{ \omega \in Y \,/\, |\omega_x| \leq \|x\|\ \forall x \in E \}$ and $K_2 := \{ \omega \in Y \,/\, \omega_{x+y} = \omega_x + \omega_y,\ \omega_{\lambda x} = \lambda \omega_x\ \forall x \in E \}$

(ENOUGH TO SHOW) WTS. K_1 is compact and K_2 is closed.

- The set $K_1 = \prod_{x\in E} [-\|x\|, \|x\|]$ $\therefore$ is COMPACT by Tychonoff's theorem.
- For K_2 first note K_2 is closed in Y since for each $\lambda \in \mathbb{R}$, $x, y \in E$ the sets.

$$A_{xy} := \{\omega \in Y;\ \omega_{x+y} - \omega_x - \omega_y = 0\}$$

$$B_{\lambda x} := \{\omega \in Y;\ \omega_{\lambda x} - \lambda \omega_x = 0\}$$

are closed in Y (maps $\omega \longmapsto \omega_{x+y} - \omega_x - \omega_y$

$\omega \longmapsto \omega_{\lambda x} - \lambda \omega_x$

are continuous) and we can write

$$K_2 := \left[\bigcap_{x,y\in E} A_{xy}\right] \cap \left[\bigcap_{\substack{x\in E\\ \lambda\in\mathbb{R}}} B_{\lambda,x}\right] \quad \underline{\underline{\text{closed}}}$$

Then K is compact being the intersection of a compact and a closed set. #

One can also prove that if E is <u>separable</u> the closed unit ball in E^* is <u>SEQUENTIALLY WEAK* COMPACT</u>

<u>i.e.</u> ∀ sequence $(f_n^*)_{n\in\mathbb{N}} \subset B_{E^*}$ contains a subsequence which converges to a point $f^* \in B_{E^*}$ in the weak* topology.