III: Banach Spaces

Reductio ad absurdum is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.

G. H. Hardy

III.1 Definition and examples †

We defined normed linear spaces in Section I.2. Since normed linear spaces are metric spaces, they may have the property of being complete.

Definition A complete normed linear space is called a **Banach space.**

Banach spaces have many of the properties of $\mathbb{R}^n$: they are vector spaces, they have a notion of distance provided by the norm, and every Cauchy sequence has a limit. In general the norm does not arise from an inner product (see Problem 4 of Chapter II), so Banach spaces are not necessarily Hilbert spaces and will not have all of the same nice geometrical properties. In order to acquaint the reader with the types of Banach spaces he is likely to encounter, we discuss several examples in detail.

Example 1 ($L^\infty(\mathbb{R})$ and its subspaces) Let $L^\infty(\mathbb{R})$ be the set of (equivalence classes of) complex-valued measurable functions on $\mathbb{R}$ such that $|f(x)| \leq M$ a.e. with respect to Lebesgue measure for some $M < \infty$ ($f \sim g$ means $f(x) = g(x)$ a.e.). Let $\|f\|_\infty$ be the smallest such M. It is an easy exercise (Problem 1) to

† A supplement to this section begins on p. 348.

show that $L^\infty(\mathbb{R})$ is a Banach space with norm $\|\cdot\|_\infty$. The bounded continuous functions $C(\mathbb{R})$ is a subspace of $L^\infty(\mathbb{R})$ and restricted to $C(\mathbb{R})$ the $\|\cdot\|_\infty$-norm is just the usual supremum norm under which $C(\mathbb{R})$ is complete (since the uniform limit of continuous functions is continuous). Thus, $C(\mathbb{R})$ is a closed subspace of $L^\infty(\mathbb{R})$.

Consider the set $\kappa(\mathbb{R})$ of continuous functions with compact support, that is, the continuous functions that vanish outside of some closed interval. $\kappa(\mathbb{R})$ is a normed linear space under $\|\cdot\|_\infty$ but is not complete. The completion of $\kappa(\mathbb{R})$ is not all of $C(\mathbb{R})$; for example, if f is the function which is identically equal to one, then f cannot be approximated by a function in $\kappa(\mathbb{R})$ since $\|f-g\|_\infty \geq 1$ for all $g \in \kappa(\mathbb{R})$. The completion of $\kappa(\mathbb{R})$ is just $C_\infty(\mathbb{R})$, the continuous functions which approach zero at $\pm\infty$ (Problem 5). Some of the most powerful theorems in functional analysis (Riesz–Markov, Stone–Weierstrass) are generalizations of properties of $C(\mathbb{R})$ (see Sections IV.3 and IV.4).

Example 2 (L^p spaces) Let $\langle X, \mu\rangle$ be a measure space and $p \geq 1$. We denote by $L^p(X, d\mu)$ the set of equivalence classes of measurable functions which satisfy:

$$\|f\|_p \equiv \left(\int_X |f(x)|^p \, d\mu(x)\right)^{1/p} < \infty$$

Two functions are equivalent if they differ only on a set of measure zero. The following theorem collects many of the standard facts about L^p spaces.

Theorem III.1 Let $1 \leq p < \infty$, then

(a) (the Minkowski inequality) If $f, g \in L^p(X, d\mu)$, then

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

(b) (Riesz–Fisher) $L^p(X, d\mu)$ is complete.

(c) (the Hölder inequality) Let p, q, and r be positive numbers satisfying $p, q, r \geq 1$ and $p^{-1} + q^{-1} = r^{-1}$. Suppose $f \in L^p(X, d\mu)$, $g \in L^q(X, d\mu)$. Then $fg \in L^r(X, d\mu)$ and

$$\|fg\|_r \leq \|f\|_p \, \|g\|_q$$

Proofs of many of the basic facts about L^p spaces, including these inequalities, can be found in the second supplemental section. The Minkowski inequality shows that $L^p(X, d\mu)$ is a vector space and that $\|\cdot\|_p$ satisfies the triangle inequality. Combined with (b) this shows that $L^p(X, d\mu)$ is a Banach space. We have given the proof of (b) for the case where $p = 1$, $X = \mathbb{R}$ and $\mu =$ Lebesgue measure; the proof for the general case is similar.

Example 3 (sequence spaces) There is a nice class of spaces which is easy to describe and which we will often use to illustrate various concepts. In the following definitions,

$$a = \{a_n\}_{n=1}^{\infty}$$

always denotes a sequence of complex numbers.

$$\ell_\infty = \left\{a \,\middle|\, \|a\|_\infty \equiv \sup_n |a_n| < \infty\right\}$$

$$c_0 = \left\{a \,\middle|\, \lim_{n\to\infty} a_n = 0\right\}$$

$$\ell_p = \left\{a \,\middle|\, |a|_p \equiv \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} < \infty\right\}$$

$$s = \left\{a \,\middle|\, \lim_{n\to\infty} n^p a_n = 0 \quad \text{for all positive integers } p\right\}$$

$$f = \left\{a \,\middle|\, a_n = 0 \quad \text{for all but a finite number of } n\right\}$$

It is clear that as sets $f \subset s \subset \ell_p \subset c_0 \subset \ell_\infty$.

The spaces ℓ_∞ and c_0 are Banach spaces with the $\|\cdot\|_\infty$ norm; ℓ_p is a Banach space with the $\|\cdot\|_p$ norm (note that this follows from Example 2 since $\ell_p = L^p(\mathbb{R}, d\mu)$ where μ is the measure with mass one at each positive integer and zero everywhere else). It will turn out that s is a Fréchet space (Section V.2). One of the reasons that these spaces are easy to handle is that f is dense in ℓ_p (in $\|\cdot\|_p$; $p < \infty$) and is dense in c_0 (in the $\|\cdot\|_\infty$ norm). Actually, the set of elements of f with only rational entries is also dense in ℓ_p and c_0. Since this set is countable, ℓ_p and c_0 are separable. ℓ_∞ is not separable (Problem 2).

Example 4 (the bounded operators) In Section I.3 we defined the concept of a bounded linear transformation or bounded operator from one normed linear space, X, to another Y; we will denote the set of all bounded linear operators from X to Y by $\mathscr{L}(X, Y)$. We can introduce a norm on $\mathscr{L}(X, Y)$ by defining

$$\|A\| = \sup_{x\in X,\, x\neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$$

This norm is often called the **operator norm.**

Theorem III.2 If Y is complete, $\mathscr{L}(X, Y)$ is a Banach space.

Proof Since any finite linear combination of bounded operators is again a bounded operator, $\mathscr{L}(X, Y)$ is a vector space. It is easy to see that $\|\cdot\|$ is a norm; for example, the triangle inequality is proven by the computation

$$\begin{aligned}\|A + B\| &= \sup_{x \neq 0} \frac{\|(A + B)x\|}{\|x\|} \leq \sup_{x \neq 0} \frac{\|Ax\| + \|Bx\|}{\|x\|} \\ &\leq \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} + \sup_{x \neq 0} \frac{\|Bx\|}{\|x\|} \\ &= \|A\| + \|B\|\end{aligned}$$

To show that $\mathscr{L}(X, Y)$ is complete, we must prove that if $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the operator norm, then there is a bounded linear operator A so that $\|A_n - A\| \to 0$. Let $\{A_n\}_{n=1}^{\infty}$ be Cauchy in the operator norm; we construct A as follows. For each $x \in X$, $\{A_n x\}_{n=1}^{\infty}$ is a Cauchy sequence in Y. Since Y is complete, $A_n x$ converges to an element $y \in Y$. Define $Ax = y$. It is easy to check that A is a linear operator. From the triangle inequality it follows that

$$| \|A_n\| - \|A_m\| | \leq \|A_n - A_m\|$$

so $\{\|A_n\|\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers converging to some real number C. Thus,

$$\begin{aligned}\|Ax\|_Y &= \lim_{n \to \infty} \|A_n x\|_Y \leq \lim_{n \to \infty} \|A_n\| \, \|x\|_X \\ &= C\|x\|_X\end{aligned}$$

so A is a bounded linear operator. We must still show that $A_n \to A$ in the operator norm. Since $\|(A - A_n)x\| = \lim_{m \to \infty} \|(A_m - A_n)x\|$, we have

$$\frac{\|(A - A_n)x\|}{\|x\|} \leq \lim_{n \to \infty} \|A_m - A_n\|$$

which implies

$$\|A - A_n\| = \sup_{x \neq 0} \frac{\|(A - A_n)x\|}{\|x\|} \leq \lim_{m \to \infty} \|A_m - A_n\|$$

which is arbitrarily small for n large enough. The triangle inequality shows that the norm of A is actually equal to C. ∎

It is important to have criteria to determine whether normed linear spaces are complete. Such a criterion is given by the following theorem (whose proof is left to Problem 3). A sequence of elements $\{x_n\}_{n=1}^{\infty}$ in a normed linear space X is called **absolutely summable** if $\sum_{n=1}^{\infty} \|x_n\| < \infty$. It is called **summable** if $\sum_{n=1}^{N} x_n$ converges as $N \to \infty$ to an $x \in X$.

Theorem III.3 A normed linear space is complete if and only if every absolutely summable sequence is summable.

For a typical application of this theorem, see the construction of quotient spaces in Section III.4. We conclude this introductory section with some definitions.

Definition A bounded linear operator from a normed linear space X to a normed linear space Y is called an **isomorphism** if it is a bijection which is continuous and which has a continuous inverse. If it is norm preserving, it is called an **isometric isomorphism** (any norm preserving map is called an isometry).

For example, we proved in Section II.3 that all separable, infinite-dimensional Hilbert spaces are isometric to ℓ_2. Two Banach spaces which are isometric can be regarded as the same as far as their Banach space properties are concerned.

We will often encounter a situation in which we have two different norms on a normed linear space.

Definition Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a normed linear space X are called **equivalent** if there are positive constants C and C' such that, for all $x \in X$,

$$C\|x\|_1 \leq \|x\|_2 \leq C'\|x\|_1$$

For example, the following three norms on $\mathbb{R}^2$ are all equivalent:

$$\|\langle x, y\rangle\|_2 = \sqrt{|x|^2 + |y|^2}$$
$$\|\langle x, y\rangle\|_1 = |x| + |y|$$
$$\|\langle x, y\rangle\|_\infty = \max\{|x|, |y|\}$$

In fact, all norms on $\mathbb{R}^2$ are equivalent; see Problem 4. The usual situation we will encounter is an incomplete normed linear space with two norms. The completions of the space in the two norms will be isomorphic if and only if the norms are equivalent. An example is provided by the sequence spaces of Example 3. The completion of f in the $\|\cdot\|_\infty$ norm is c_0 while the completion in the $\|\cdot\|_p$ norm is ℓ_p. Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a normed linear space X are equivalent if and only if the identity map is an isomorphism from $\langle X, \|\cdot\|_1\rangle$ to $\langle X, \|\cdot\|_2\rangle$.

In the last section we proved that the set of bounded linear transformations from one Banach space X to another Y was itself a Banach space. In the case where Y is the complex numbers, this space $\mathscr{L}(X, \mathbb{C})$ is denoted by X^* and called the **dual space** of X. The elements of X^* are called bounded linear functionals on X. In this chapter when we talk about convergence in X^* we always mean convergence in the norm given in Theorem III.2. If $\lambda \in X^*$, then

$$\|\lambda\| = \sup_{x \in X,\ \|x\| \le 1} |\lambda(x)|$$

In Section IV.5, we discuss another notion of convergence for X^*.

Dual spaces play an important role in mathematical physics. In many models of physical systems, whether in quantum mechanics, statistical mechanics, or quantum field theory, the possible states of the system in question can be associated with linear functionals on appropriate Banach spaces. Furthermore, linear functionals are important in the modern theory of partial differential equations. For these reasons, and because they are interesting in their own right, dual spaces have been studied extensively. There are two directions in which such study can proceed: either determining the dual spaces of particular Banach spaces or proving general theorems relating properties of Banach spaces to properties of their duals. In this section we study several examples of special interest and prove one general theorem. For an example of another general theorem see Theorem III.7.

Example 1 (L^p spaces) Suppose that $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ then, according to the Hölder inequality (Theorem III.1), fg is in $L^1(\mathbb{R})$. Thus,

$$\int_{-\infty}^{\infty} \overline{g(x)} f(x)\, dx$$

makes sense. Let $g \in L^q(\mathbb{R})$ be fixed and define

$$G(f) = \int_{-\infty}^{\infty} \bar{g} f\, dx$$

for each $f \in L^p(\mathbb{R})$. The Hölder inequality shows that $G(\cdot)$ is a bounded linear functional on $L^p(\mathbb{R})$ with norm less than or equal to $\|g\|_q$; actually the norm is equal to $\|g\|_q$. The converse of this statement is also true. That is, every bounded linear functional on L^p is of the form $G(\cdot)$ for some $g \in L^q$. Furthermore, different functions in L^q give rise to different functionals on L^p. Thus, the mapping that assigns to each $g \in L^q$ the corresponding linear functional, $G(\cdot)$, on $L^p(\mathbb{R})$ is a (conjugate linear) isometric isomorphism of L^q onto $(L^p)^*$. In this sense, L^q is the dual of L^p. Since the roles of p and q in the expression $p^{-1}+q^{-1}=1$ are symmetric, it is clear that $L^p=(L^q)^*=((L^p)^*)^*$. That is, the dual of the dual of L^p is again L^p.

The case where $p=1$ is different. The dual of $L^1(\mathbb{R})$ is $L^\infty(\mathbb{R})$ with the elements of $L^\infty(\mathbb{R})$ acting on functions in $L^1(\mathbb{R})$ in the natural way given by the above integral. However, the dual of $L^\infty(\mathbb{R})$ is not $L^1(\mathbb{R})$ but a much larger space (see Problems 7 and 8). As a matter of fact, we will prove later (Chapter XVI) that $L^1(\mathbb{R})$ is not the dual of any Banach space. The duality statements in this example hold for $L^p(X, d\mu)$ where $\langle X, \mu\rangle$ is a general measure space except that $L^1(X)$ may be the dual of $L^\infty(X)$ if $\langle X, \mu\rangle$ is trivially small.

Example 2 (Hilbert spaces) If we let $p=2$ in Example 1, then $q=2$ and we obtain the result that $L^2(\mathbb{R})=L^2(\mathbb{R})^*$, that is, $L^2(\mathbb{R})$ is its own dual space. In fact, we have already shown (the Riesz lemma) in Section II.2 that this is true for all Hilbert spaces. The reader is cautioned again that the map which identifies $\mathscr{H}$ with its dual $\mathscr{H}^*$ is conjugate linear. If $g \in \mathscr{H}$, then the linear functional G corresponding to g is $G(f)=(g,f)$.

Example 3 ($\ell_\infty=\ell_1^*, \ell_1=c_0^*$) Suppose that $\{\lambda_k\}_{k=1}^\infty \in \ell_1$. Then for each $\{a_k\}_{k=1}^\infty \in c_0$

$$\Lambda(\{a_k\}_{k=1}^\infty)=\sum_{k=1}^\infty \lambda_k a_k$$

converges and $\Lambda(\cdot)$ is a continuous linear functional on c_0 with norm equal to $\sum_{k=1}^\infty |\lambda_k|$. To see that all continuous linear functionals on c_0 arise in this way, we proceed as follows. Suppose $\lambda \in c_0^*$ and let e^k be the sequence in c_0 which has all its terms equal to zero except for a one in the kth place. Define $\lambda_k=\lambda(e^k)$ and let $f^\ell=\sum_{k=1}^\ell(|\lambda_k|/\lambda_k)e^k$. If some λ_k is zero, we simply omit that term from the sum. Then for each $\ell, f^\ell \in c_0$ and $\|f^\ell\|_{c_0}=1$. Since,

$$\lambda(f^\ell)=\sum_{k=1}^\ell |\lambda_k| \qquad \text{and} \qquad |\lambda(f^\ell)| \le \|f^\ell\|_{c_0}\|\lambda\|_{c_0^*}$$

we have

$$\sum_{k=1}^\ell |\lambda_k| \le \|\lambda\|_{c_0^*}$$

Since this is true for all ℓ, $\sum_{k=1}^\infty |\lambda_k|<\infty$ and

$$\Lambda(\{a_k\}_{k=1}^\infty)=\sum_{k=1}^\infty \lambda_k a_k$$

is a well-defined linear functional on c_0. However, $\lambda(\cdot)$ and $\Lambda(\cdot)$ agree on finite linear combinations of the e_k. Because such finite linear combinations are dense in c_0, we conclude that $\lambda = \Lambda$. Thus every functional in c_0^* arises from a sequence in ℓ_1, and the reader can check for himself that the norms in ℓ_1 and c_0^* coincide. Thus $\ell_1 = c_0^*$. A similar proof shows that $\ell_\infty = \ell_1^*$.

Since the dual X^* of a Banach space is itself a Banach space (Theorem III.2), it also has a dual space, denoted by X^{**}. X^{**} is called the **second dual**, the **bidual**, or the **double dual** of the space X. In Example 3, ℓ_1 is the first dual of c_0 and ℓ_∞ is the second dual. It is not a priori evident that X^* is always nonzero and if $X^* = \{0\}$ then $X^{**} = \{0\}$ too. However, this situation does not occur; dual spaces always have plenty of linear functionals in them. We prove this fact in the next section. Using a corollary also proven there we will prove that X can be regarded in a natural way as a subset of X^{**}.

Theorem III.4 Let X be a Banach space. For each $x \in X$, let $\tilde{x}(\cdot)$ be the linear functional on X^* which assigns to each $\lambda \in X^*$ the number $\lambda(x)$. Then the map $J: x \to \tilde{x}$ is an isometric isomorphism of X onto a (possibly proper) subspace of X^{**}.

Proof Since

$$|\tilde{x}(\lambda)| = |\lambda(x)| \le \|\lambda\|_{X^*}\|x\|_X$$

$\tilde{x}$ is a bounded linear functional on X^* with norm $\|\tilde{x}\|_{X^{**}} \le \|x\|_X$. It follows from Theorems III.5 and III.6 that, given x, we can find a $\lambda \in X^*$ so that

$$\|\lambda\|_{X^*} = 1 \qquad \text{and} \qquad \lambda(x) = \|x\|_X$$

This shows that

$$\|\tilde{x}\|_{X^{**}} = \sup_{\lambda \in X^*,\ \|\lambda\| \le 1} |\tilde{x}(\lambda)| \ge \|x\|_X$$

which implies that

$$\|\tilde{x}\|_{X^{**}} = \|x\|_X$$

Thus, J is an isometry of X into X^{**}. ∎

Definition If the map J, defined in Theorem III.4, is surjective, then X is said to be **reflexive**.

The $L^p(\mathbb{R})$ spaces are reflexive for $1 < p < \infty$ since $(L^p)^{**} = (L^q)^* = L^p$, but $L^1(\mathbb{R})$ is not reflexive. All Hilbert spaces are reflexive. c_0 is not reflexive, since its double dual is ℓ_∞. The theory of reflexive spaces is developed further in Problems 22 and 26 of this chapter and Problem 15 of Chapter V.

III.3 The Hahn–Banach theorem

In dealing with Banach spaces, one often needs to construct linear functionals with certain properties. This is usually done in two steps: first one defines the linear functional on a subspace of the Banach space where it is easy to verify the desired properties; second, one appeals to (or proves) a general theorem which says that any such functional can be extended to the whole space while retaining the desired properties. One of the basic tools of the second step is the following theorem, whose variants will reappear in Section V.1 and Chapter XIV.

Theorem III.5 (Hahn–Banach theorem) Let X be a real vector space, p a real-valued function defined on X satisfying $p(\alpha x + (1-\alpha)y) \leq \alpha p(x) + (1-\alpha)p(y)$ for all x and y in X and all $\alpha \in [0, 1]$. Suppose that λ is a linear functional defined on a subspace Y of X which satisfies $\lambda(x) \leq p(x)$ for all $x \in Y$. Then, there is a linear functional Λ, defined on X, satisfying $\Lambda(x) \leq p(x)$ for all $x \in X$, such that $\Lambda(x) = \lambda(x)$ for all $x \in Y$.

Proof The idea of the proof is the following. First we will show that if $z \in X$ but $z \notin Y$, then we can extend λ to a functional having the right properties on the space spanned by z and Y. We then use a Zorn's lemma argument to show that this process can be continued to extend λ to the whole space X.

Let $\tilde{Y}$ denote the subspace spanned by Y and z. The extension of λ to Y, call it $\tilde{\lambda}$, is specified as soon as we define $\tilde{\lambda}(z)$ since

$$\tilde{\lambda}(az + y) = a\tilde{\lambda}(z) + \lambda(y)$$

Suppose that $y_1, y_2 \in Y$, $\alpha, \beta > 0$. Then

$$\begin{aligned}
\beta\lambda(y_1) + \alpha\lambda(y_2) = \lambda(\beta y_1 + \alpha y_2) &= (\alpha+\beta)\lambda\left(\frac{\beta}{\alpha+\beta}y_1 + \frac{\alpha}{\alpha+\beta}y_2\right) \\
&\leq (\alpha+\beta)p\left(\frac{\beta}{\alpha+\beta}(y_1 - \alpha z) + \frac{\alpha}{\alpha+\beta}(y_2 + \beta z)\right) \\
&\leq \beta p(y_1 - \alpha z) + \alpha p(y_2 + \beta z)
\end{aligned}$$

Thus, for all $\alpha, \beta > 0$ and $y_1, y_2 \in Y$,

$$\frac{1}{\alpha}[-p(y_1 - \alpha z) + \lambda(y_1)] \leq \frac{1}{\beta}[p(y_2 + \beta z) - \lambda(y_2)]$$

We can therefore find a real number a such that

$$\sup_{\substack{y \in Y \\ \alpha > 0}} \left[\frac{1}{\alpha}(-p(y - \alpha z) + \lambda(y))\right] \leq a \leq \inf_{\substack{y \in Y \\ \alpha > 0}} \left[\frac{1}{\alpha}(p(y + \alpha z) - \lambda(y))\right]$$

We now define $\tilde{\lambda}(z) = a$. It may be easily verified that the resulting extension satisfies $\tilde{\lambda}(x) \leq p(x)$ for all $x \in \tilde{Y}$. This shows that λ can be extended one dimension at a time.

We now proceed with the Zorn's lemma argument. Let $\mathscr{E}$ be the collection of extensions e of λ which satisfy $e(x) \leq p(x)$ on the subspace where they are defined. We partially order $\mathscr{E}$ by setting $e_1 \prec e_2$ if e_2 is defined on a larger set than e_1 and $e_2(x) = e_1(x)$ where they are both defined. Let $\{e_\alpha\}_{\alpha \in A}$ be a linearly ordered subset of $\mathscr{E}$; let X_α be the subspace on which e_α is defined. Define e on $\bigcup_{\alpha \in A} X_\alpha$ by setting $e(x) = e_\alpha(x)$ if $x \in X_\alpha$. Clearly $e_\alpha \prec e$ so each linearly ordered subset of $\mathscr{E}$ has an upper bound. By Zorn's lemma, $\mathscr{E}$ has a maximal element Λ, defined on some set X', satisfying $\Lambda(x) \leq p(x)$ for $x \in X'$. But, X' must be all of X, since otherwise we could extend Λ to a $\tilde{\Lambda}$ on a larger space by adding one dimension as above. Since this contradicts the maximality of Λ, we must have $X = X'$. Thus, the extension Λ is everywhere defined. ∎

In the theorem we have just proven, X is a real vector space. We now extend the theorem to the case where X is complex.

Theorem III.6 (complex Hahn–Banach theorem) Let X be a complex vector space, p a real-valued function defined on X satisfying $p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y)$ for all $x, y \in X$, and $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| = 1$. Let λ be a complex linear functional defined on a subspace Y of X satisfying $|\lambda(x)| \leq p(x)$ for all $x \in Y$. Then, there exists a complex linear functional Λ, defined on X, satisfying $|\Lambda(x)| \leq p(x)$ for all $x \in X$ and $\Lambda(x) = \lambda(x)$ for all $x \in Y$.

Proof Let $\ell(x) = \mathrm{Re}\{\lambda(x)\}$. ℓ is a real linear functional on Y and since

$$\ell(ix) = \mathrm{Re}\{\lambda(ix)\} = \mathrm{Re}\{i\lambda(x)\} = -\mathrm{Im}\{\lambda(x)\}$$

we see that $\lambda(x) = \ell(x) - i\ell(ix)$. Since ℓ is real linear and $p(\alpha x + (1 - \alpha)y) \leq \alpha p(x) + (1 - \alpha)p(y)$ for $\alpha \in [0, 1]$, ℓ has a real linear extension L to all of X obeying $L(x) \leq p(x)$ (by Theorem III.5). Define $\Lambda(x) = L(x) - iL(ix)$. Λ clearly extends λ and is real linear. Moreover, $\Lambda(ix) = L(ix) - iL(-x) = i\Lambda(x)$, so Λ is also complex linear. To complete the proof, we need only show that $|\Lambda(x)| \le p(x)$. First, note that $p(\alpha x) = p(x)$ if $|\alpha| = 1$. If we let $\theta = \operatorname{Arg}\{\Lambda(x)\}$ and use the fact that $\operatorname{Re}\Lambda = L$, we see that

$$\begin{aligned}|\Lambda(x)| &= e^{-i\theta}\Lambda(x) = \Lambda(e^{-i\theta}x) = L(e^{-i\theta}x)\\ &\le p(e^{-i\theta}x) = p(x) \quad \blacksquare\end{aligned}$$

Corollary 1 Let X be a normed linear space, Y a subspace of X, and λ an element of Y^*. Then there exists a $\Lambda \in X^*$ extending λ and satisfying $\|\Lambda\|_{X^*} = \|\lambda\|_{Y^*}$.

Proof Choose $p(x) = \|\lambda\|_{Y^*}\|x\|$ and apply the above theorems. ∎

Corollary 2 Let y be an element of a normed linear space X. Then there is a nonzero $\Lambda \in X^*$ such that $\Lambda(y) = \|\Lambda\|_{X^*}\|y\|$.

Proof Let Y be the subspace consisting of all scalar multiples of y and define $\lambda(ay) = a\|y\|$. By using Corollary 1, we can construct Λ with $\|\Lambda\| = \|\lambda\|$ extending λ to all of X. But, since $\Lambda(y) = \|y\|$, $\|\Lambda\| = 1$ and therefore

$$\Lambda(y) = \|\Lambda\|_{X^*}\|y\| \quad \blacksquare$$

Corollary 3 Let Z be a subspace of a normed linear space X and suppose that y is an element of X whose distance from Z is d. Then there exists a $\Lambda \in X^*$ so that $\|\Lambda\| \le 1$, $\Lambda(y) = d$, and $\Lambda(z) = 0$ for all z in Z.

The proof of the third corollary is left to the reader (Problem 10). To show how useful these corollaries are we prove the following general theorem.

Theorem III.7 Let X be a Banach space. If X^* is separable, then X is separable.

Proof Let $\{\lambda_n\}$ be a dense set in X^*. Choose $x_n \in X$, $\|x_n\| = 1$, so that

$$|\lambda_n(x_n)| \ge \|\lambda_n\|/2$$

Let $\mathcal{D}$ be the set of all finite linear combinations of the $\{x_n\}$ with rational coefficients. Since $\mathcal{D}$ is countable, it is sufficient to show that $\mathcal{D}$ is dense in X. If $\mathcal{D}$ is not dense in X, then there is a $y \in X\backslash\mathcal{D}$ and a linear functional $\lambda \in X^*$ so that $\lambda(y) \ne 0$, but $\lambda(x) = 0$ for all $x \in \mathcal{D}$ (Corollary 3). Let $\{\lambda_{n_k}\}$ be a subsequence of $\{\lambda_n\}$ which converges to λ. Then

$$\begin{aligned}\|\lambda - \lambda_{n_k}\|_{X^*} &\ge |(\lambda - \lambda_{n_k})(x_{n_k})|\\ &= |\lambda_{n_k}(x_{n_k})| \ge \|\lambda_{n_k}\|/2\end{aligned}$$

which implies $\|\lambda_{n_k}\| \to 0$ as $k \to \infty$. Thus $\lambda = 0$ which is a contradiction. Therefore $\mathscr{D}$ is dense and X is separable. ∎

The example of ℓ_1 and ℓ_∞ shows that the converse of this theorem does not hold. Incidently, Theorem III.7 provides a proof that ℓ_1 is not the dual of ℓ_∞, since ℓ_1 is separable and ℓ_∞ is not.

III.4 Operations on Banach spaces

We have already seen several ways in which new Banach spaces can arise from old ones. The successive duals of a Banach space are Banach spaces and the bounded operators from one Banach space to another form a Banach space. Also, any closed linear subspace of a Banach space is a Banach space. There are two other ways of constructing new Banach spaces which we will need: direct sums and quotient spaces.

Let A be an index set (not necessarily countable), and suppose that for each $\alpha \in A$, X_α is a Banach space. Let

$$X = \{\{x_\alpha\}_{\alpha \in A} \mid x_\alpha \in X_\alpha,\ \sum_{\alpha \in A} \|x_\alpha\|_{X_\alpha} < \infty\}$$

Then X with the norm

$$\|\{x_\alpha\}\| = \sum_{\alpha \in A} \|x_\alpha\|_{X_\alpha}$$

is a Banach space. It is called the **direct sum** of the spaces X_α and is often written $X = \bigoplus_{\alpha \in A} X_\alpha$. We remark that the Hilbert space direct sum and the Banach space direct sum are not necessarily the same. For example, if we take a countable number of copies of $\mathbb{C}$, the Banach space direct sum is ℓ_1, while the Hilbert space direct sum is ℓ_2. However, if one has a *finite* number of Hilbert spaces, their Hilbert space direct sum and their Banach space direct sum are isomorphic in the sense of Section III.1.

Let M be a closed linear subspace of a Banach space X. If X were a Hilbert space, we could write $X = M \oplus M^\perp$. The Banach space that we now define can sometimes take the place of $M^\perp$ in the Banach space case where there is no orthogonality. If x and y are elements of X, we will write $x \sim y$ if $x - y \in M$. The relation $\sim$ is an equivalence relation; we denote the set of equivalence classes by X/M. As usual we denote the equivalence class containing x by $[x]$. We define addition and scalar multiplication of equivalence classes by

$$\alpha[x] + \beta[y] = [\alpha x + \beta y]$$