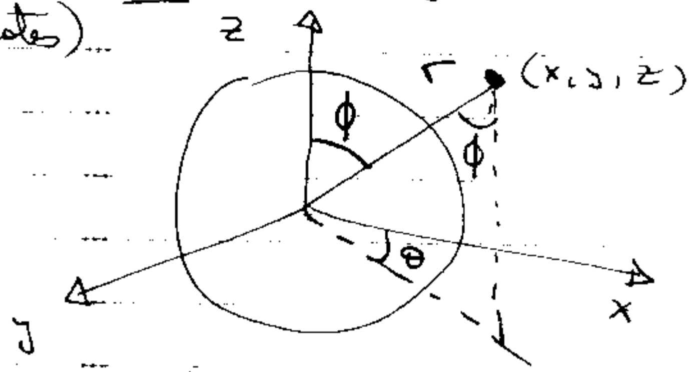
Folland §2.7

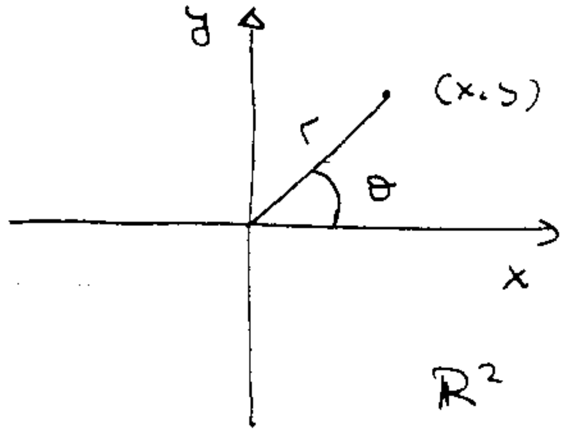
Math 624 Handout on the Lebesgue measure on $\mathbb{R}^n$: Polar coordinates in $\mathbb{R}^n$

(polar coord.) $\mathbb{R}^2$: $(x, y) \longrightarrow (r\cos\theta,\ r\sin\theta)$

(spherical coordinates) $\mathbb{R}^3$: $(x, y, z) \longrightarrow (r\sin\phi\cos\theta,\ r\sin\phi\sin\theta,\ r\cos\phi)$



$\mathbb{R}^3$



$\mathbb{R}^2$

$dx\,dy\,dz = r^2 \sin\phi\, dr\, d\theta\, d\phi$ $\qquad$ $dx\,dy = r\,dr\,d\theta$

In general, for $\mathbb{R}^n$ precise formulas become very complex but for INTEGRATION theory what's important is that if λ = n-dim Lebesgue msr. then $\boxed{d\lambda \rightarrow r^{n-1}\, d\sigma(x')\, dr}$

where $x' = \dfrac{x}{|x|} \in S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$

$$\begin{cases} r^{n-1}dr & \text{is measure on } (0,\infty) \\ d\sigma(x') & \text{is a ``surface measure'' on } S^{n-1} \end{cases}$$

Polar coordinates are defined then via a nonlinear transformation (continuous BIJECTION)

$$\psi : \mathbb{R}^n \setminus \{0\} \longrightarrow (0,\infty) \times S^{n-1}$$

$$x \longrightarrow (r, x') \qquad x' = \frac{x}{|x|}$$

whose continuous inverse $\qquad r \in (0,\infty)$

is the map $\psi^{-1}(r, x') = r \cdot x'$

Denote by m_* = Borel measure on $(0,\infty) \times S^{n-1}$ induced by ψ from the Lebesgue measure on $\mathbb{R}^n$

That is: $\quad m_*(E) = \lambda(\psi^{-1}(E))$

Define also ρ on $(0,\infty)$ by

$$\rho(E) = \int_E r^{n-1}\, dr$$

THEOREM: There is a unique Borel measure $\sigma = \sigma_{n-1}$ on S^{n-1} such that $m_* = \rho \times \sigma$

If f is Borel measurable on $\mathbb{R}^n$, $f \in L^+(\lambda)$ or $f \in L^1(\lambda)$

then (*) $\quad \int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_{S^{n-1}} f(rx')\, r^{n-1}\, d\sigma(x')\, dr$

<u>Proof</u>: To show (*) note that when f is the characteristic function of a set it follows from the definition of $m_* = \rho \times \sigma$. For general f follows then as usual by linearity and approximation arguments. (CHECK !!)

We concentrate then in the <u>construction of σ</u>:

Let $\mathbf{B}$ be a Borel set <u>in S^{n-1}</u>, for $a>0$ let

$$B_a = \varphi^{-1}\big((0,a]\times B\big)$$
$$= \{ r x' : 0 < r \le a \,,\ x' \in B \}$$

If (*) is to hold for $f = \chi_{\mathbf{B}_1}$ then we must have that

$$\lambda(\mathbf{B}_1) = \int_0^1 \int_{\mathbf{B}} r^{n-1}\, d\sigma(x')\, dr$$
$$= \sigma(B) \int_0^1 r^{n-1}\, dr = \frac{\sigma(B)}{n}$$

Thus we DEFINE $\sigma(B) = n \cdot \lambda(B_1)$

Since the map $B \longmapsto B_1$ sends Borel sets into Borel sets and commutes with countable unions and complements we get that σ is indeed a Borel measure on S^{n-1}

By dilation $B_1 \to B_a$ and since this map $x \longrightarrow a \cdot x$ is in $GL(n,\mathbb{R})$ we get that $\lambda(B_a) = a^n \lambda(B_1)$ and hence for $0 < a < b$

$$
\begin{aligned}
m_*\big((a,b] \times B\big) &= \lambda(B_b \setminus B_a) \\
&= \frac{b^n - a^n}{n}\,\sigma(B) \\
&= \sigma(B) \int_a^b r^{n-1}\,dr \\
&= \rho \times \sigma\big((a,b] \times B\big)
\end{aligned}
$$

To finish, note that if we fix B a Borel set in S^{n-1} and let $\mathcal{A}_B$ be the collection of finite disjoint unions of sets of the form $(a,b] \times B$ (elementary family) then $\mathcal{A}_B$ is an algebra on $(0,\infty) \times B$ that generates the σ-algebra $\mathcal{M}_B = \{A \times B :$ $A \in$ Borel sets in $(0,\infty)\}$.

By the calculation above we have that

$m_* = \rho \times \sigma$ on $\mathcal{M}_B$.

But the union of all $\mathcal{M}_B$ as B ranges over S^{n-1} coincides with the set of Borel rectangles in $(0,\infty) \times S^{n-1}$.

By uniqueness then we must have $m_* = \rho \times \sigma$ on all Borel sets. #

Remark: By considering the completion of the measure σ (*) can be extended to LEBESGUE measurable functions as well.

COROLLARY 1 If f is a measurable fc. on $\mathbb{R}^n$ non negative or integrable ($f \in L^+(\lambda)$ or $f \in L^1(\lambda)$) such that $f(x) = g(|x|)$ for some g on $(0,\infty)$ (i.e. f is radial function) then

$$\int_{\mathbb{R}^n} f(x)\,dx = \sigma(S^{n-1}) \int_0^\infty g(r)\, r^{n-1}\, dr$$

COROLLARY 2 Let $c, C > 0$ and let

$B = \{ x \in \mathbb{R}^n : |x| < c \}$.

Let f be a measurable fc. on $\mathbb{R}^n$.

(a) If $|f(x)| \le C|x|^{-a}$ on B for some $a < n$ then $f \in L^1(B)$.

However if $|f(x)| \ge C|x|^{-n}$ on B then $f \notin L^1(B)$

(b) If $|f(x)| \le C|x|^{-a}$ on B^c for some $a > n$ then $f \in L^1(B^c)$.

However if $|f(x)| \ge C|x|^{-n}$ on B^c then $f \notin L^1(B^c)$

Proof: for (a) Consider $\tilde{f} = |x|^{-a} \chi_B$ and apply the previous Corollary 1 to $\tilde{f}$.

For (b) consider $\tilde{f} = |x|^{-a} \chi_{B^c}$ and apply the previous Corollary 1 to $\tilde{f}$.

In both cases note $\tilde{f}$ is RADIAL. #

Question: What is $\sigma(S^{n-1})$?

Answer: $\sigma(S^{n-1}) = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$

where Γ = gamma function defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad (\operatorname{Re} z > 0)$$

$$\Gamma(n) = (n-1)! \quad \text{and} \quad \Gamma\left(\tfrac{1}{2}n\right) = \frac{\Gamma\left(\frac{1}{2}n + 1\right)}{\frac{1}{2}n}.$$

To prove this we first need a calculation:

Lemma: For $a > 0$

$$\int_{\mathbb{R}^n} \exp\left(-a|x|^2\right) dx = \left(\frac{\pi}{a}\right)^{n/2}$$

Proof: Denote by $I_n = \int_{\mathbb{R}^n} \exp(-a|x|^2)$

If $n=2$ we have by Corollary 1

$$I_2 = 2\pi \int_0^\infty r e^{-ar^2} dr = -\frac{\pi}{a} e^{-ar^2}\Big|_0^\infty$$

$$= \boxed{\frac{\pi}{a}}$$

Now note that

$$\exp(-a|x|^2) = \prod_{j=1}^{n} \exp(-a x_j^2)$$

By invoking Tonelli's theorem ($\exp(-a x_j^2) \in L^+$)

we get that

$$I_n = (I_1)^n$$

In particular that $I_2 = I_1^2 \Rightarrow$

$$I_1 = \left(\frac{\pi}{a}\right)^{1/2} \Rightarrow \boxed{I_n = \left(\frac{\pi}{a}\right)^{n/2}}$$

Now: $\pi^{n/2} = \int_{\mathbb{R}^n} e^{-|x|^2} dx =$

$$= \sigma(S^{n-1}) \int_0^\infty r^{n-1} e^{-r^2}\, dr$$

$$= \frac{\sigma(S^{n-1})}{2} \int_0^\infty s^{n/2} e^{-s} \frac{ds}{s}$$

substitute

$\boxed{s = r^2}$

$ds = 2r\,dr$

$$= \frac{\sigma(S^{n-1})}{2} \Gamma\left(\frac{n}{2}\right)$$

<u>COROLLARY 3</u>: If $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$

then $$\lambda(B^n) = \frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n + 1)}$$

<u>Proof</u>: Note that by functional equation for Γ, $\frac{1}{2}n\, \Gamma(\frac{1}{2}n) = \Gamma(\frac{1}{2}n + 1)$

and $\lambda(B^n) = \frac{1}{n}\sigma(S^{n-1})$ by definition of σ

Hence the desired conclusion follows. #