M725–Functional Analysis
Homework– Spring 2020

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Problems listed are from Brezis's Book. Note that each problem in the book has several parts. Do alll of them unless otherwise noted.

- **Set 1. Due date: 02/13/2020**

Problems (from class):

Problem 1 Show that p is sublinear iff p is convex and $p(\alpha x) = \alpha p(x)$ for $\alpha \geq 0$

Problem 2 Let

$$P := \left\{ (Z, g) \;:\; Z \subseteq X \text{ subspace }, Y \subset Z \text{ and } g : Z \to \mathbb{R} \text{ linear, } g|_Y = f \text{ and } g \leq p \text{ on } Z \right\}$$

Endow P with the following relation: For $(Z_1, g_1), (Z_2, g_2) \in P$ let:

$$(Z_1, g_1) \leq (Z_2, g_2) \iff Z_1 \subseteq Z_2 \text{ and } g_2 = g_1 \text{ on } Z_1$$

Check that $\leq$ is a partial order in P.

Problem 3 For (Z_*, g_*) as defined in our proof of HB PartII (of an infinite-dimensional vector space), show that Z_* is a subspace and that $g_* : Z_* \to \mathbb{R}$ is linear. Hence $(Z_*, g_*) \in P$.

Problem 4 Prove Hahn-Banach Part II with X being a normed space over $\mathbb{C}$ (instead of $\mathbb{R}$), where p is the norm

Problem 5 Let X be a normed vector space (n.v.s) and let $Y \subset X$ be a subspace. Define

$$X^* := \{x^* : X \to \mathbb{R}, \text{ linear and bounded}\}$$

$$Y^* := \{y^* : Y \to \mathbb{R}, \text{ linear and bounded}\}$$

(see Brezis pp. 3) Check that the *dual norm*, defined as:

$$\|x^*\|_{X^*} := \sup_{x \in X : \|x\|_X < 1} |x^*(x)|$$

is indeed a norm. Also check that $X^* \subseteq Y^*$.

<u>Chapter 1</u> 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8.

<u>Additional Problem</u>: Let X be the vector space of all sequences $\{a_n\}_{n\geq 1}$of real numbers with only finitely many nonzero terms. Consider the ℓ^1-norm $\|\{a_n\}\|_1 = \sum_{n=1}^{\infty} |a_n|$ and the ℓ^2-norm, $\|\{a_n\}\|_2 = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}$. Prove that $\|\cdot\|_1$ and $\|\cdot\|_2$ are **not** equivalent norms.

- **Set 2. Due date: 03/26/2020**

<u>Problems (from class)</u>:

<u>Problem 1</u> Let $(E, \|\cdot\|)$ be a n.v.s., and let $A \subset E$, $B \subset E$ be nonempty convex subsets. Assume A is closed and B is compact, and $A \cap B = \varnothing$, define

$$C := A - B = \{z \in C \ : \ z = x - y, \quad x \in A, \, y \in B\}$$

Prove that C is closed.

<u>Problem 2</u> By Hahn-Banach (first geometric form), there exists a closed hyperplane that separates $B(0, r)$ and C. Then, there exists $f \in E^*$, nontrivial such that

$$f(x - y) \leq f(rz), \quad \forall x \in A, \, \forall y \in B, \, \forall z \in B(0, 1)$$

Let

$$\varepsilon := \frac{r}{2}\|f\|_{E^*} > 0$$

Show that

$$f(x) + \varepsilon \leq f(y) - \varepsilon, \quad \forall x \in A, \, \forall y \in B$$

(equivalently, you can show that $f(x - y) \leq -r\|f\|$.)

<u>Problem 3</u> Consider Corollary 1.8 (Brezis pp. 8):

Corollary Let $F \subset E$ be a linear subspace such that $\overline{F} \neq E$. Then there exists some $f \in E^*$, $f \not\equiv 0$ such that

$$\langle f, x\rangle = 0 \quad \forall x \in F$$

Remark This corollary is useful to show that a linear subspace $F \subset E$ is dense in E. Then it'd suffice to show that every continuous linear functional on E that vanishes on F must vanish everywhere on E.

Show that the remark and corollary together imply the density of F in E.

<u>Problem 4</u> Let X, Y be Banach spaces. Let $T: X \to Y$ be bounded and linear. Show that if T^* is invertible, then T^* is open.

<u>Problem 5</u> Let X, Y be Banach spaces. Let $T \in \mathcal{L}(X, Y)$ such that T^* is an injection with closed range. Prove that T is surjective.

<u>Problem 6 (Class of 02/26/2020)</u> ref. Brezis, pp 39, Theorem 2.12:
Assume that $T \in \mathcal{L}(E, F)$ surjective. The following properties are equivalent:

(1) T admits a right inverse

and

(2) $N(T) = T^{-1}(0)$ admits a complement in E

In the proof of (2) $\Longrightarrow$ (1): Let L be a complement of $N(T)$. Let P be the (continuous) projection operator from E to L. Given $f \in F$, we denote by x any solution of the equation $Tx = f$. Set $Sf = Px$ and note that S is independent of the choice of x. It is easy to check that $S \in \mathcal{L}(F, E)$ and that $T \circ S = I_F$.

Explicitly check the last statement.

<u>Problem 7 (Class of 03/03/2020)</u> In the example: $E = F = \ell^1, E^* = \ell^\infty$, consider $A \in \mathcal{L}(E, F)$ (linear, not necessarily bounded) such that $A : D(A) \to E$.

$$D(A) = \{\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \ell^1 : A(x_n)_{n\in\mathbb{N}} = n(x_n) \in \ell^1\}$$

Show that $D(A)$ is dense in ℓ^1 and A is closed.

<u>Chapter 2 </u>: 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12.

• Set 3. Due date: 04/28/2020

<u>Chapter 3</u>: 3.1, 3.3 (use HB in one direction), 3.5, 3.9, 3.10, 3.16 (1. and 2.), 3.18, 3.22

<u>Chapter 6:</u> 6.1, 6.2, 6.3, 6.7 1)2)4), 6.8, 6.11.