M534H HOMEWORK– Spring 2020

Prof. Andrea R. Nahmod

Do not turn in problems that have an * next to the number.

- **Set 1. Due date: 02/06/2020**

Section 1.1: 2, 3, 4, 10, 11, 12

Additional Work: Read carefully Appendices A1. A2 and A3.

Section 1.2: 1, 2.

Additional Problem 1: Solve the transport equation $5u_x - 6u_y = 0$ together with the auxiliary condition that $u(x, 0) = 4x^3$.

Additional Problem 2: Solve the inhomogeneous transport equation $2u_x + 3u_y = 1$.

Additional Problem 3: Solve the linear homogeneous equation $u_x + u_y + u = 0$.

Additional Problem 4: a) Check that

$$u(x, y) = \frac{1}{4}(e^{x+2y} - e^{x-2y})$$

solves the inhomogeneous equation

$$u_x + u_y + u = e^{x+2y}.$$

b) Next use the additional problem 3 to write the general form of the solutions to

$$u_x + u_y + u = e^{x+2y}.$$

c) Find the solution to $u_x + u_y + u = e^{x+2y}$ that also satisfies $u(x, 0) = 1$.

- **Set 2. Due date: 02/13/2020**

Additional Problem:

a) Find the general solution to Problem 8 in Section 1.2. Specify what method you are using and explain step by step your work. Show all your work.

b) Choose $a = 2, b = 5$ and $c = 29$ and find the solution $u(x, y)$ to part a) that also satisfies $u(x, 0) = e^{-3x}$.

c) Check that $\frac{1}{6}(e^{x+y} - e^{3x-y})$ is a particular solution to the inhomogeneous equation

$$2u_x + 5u_y + 29\,u = (6e^{x+y} - 5e^{3x-y})$$

d) Use part a) $a = 2, b = 5$ and $c = 29$ together with part c) to find the general solution to the inhomogeneous equation

$$2u_x + 5u_y + 29\,u = (6e^{x+y} - 5e^{3x-y})$$

Section 1.2: 3, 4, 5, 6.

- **Set 3. Due date: 02/27/2020**

(Do not turn in problems that have an * next to the number.)

Section 1.3: 6, 9, 10 (in $\mathbb{R}^3$), 11*.

Extra Problem* : Prove the *Second Vanishing Theorem* in A.1. page 416

Section 1.4: Read Section 1.4. Then do Problem 1.

Section 1.5: 1, 4, 5, and:

Problem 6 (modified): Solve the equation $u_x + 2x\,y^2\,u_y = 0$ and find a solution that satisfies the auxiliary condition $u(0, y) = y$.

Section 1.6: 1, 2, 4

Additional Problem: Find the regions in $\mathbb{R}^2$ where $x^2\,u_{xx} + 4u_{xy} + y^2\,u_{yy} = 0$ is respectively elliptic, parabolic, hyperbolic. Plot these regions.

- **Set 4. Due date: 03/05/2020**

Section 2.1: 1, 2, 8, 9, 10, 11

Hint for 11: $-\frac{1}{16}sin(x + t)$ is a particular solution. Check it!.

Additional Problem 1: First find the solution to the linear homogeneous wave equation with wave speed 1 and with initial conditions $u(x, 0) = \sin x$, $u_t(x, 0) = 0$. Then calculate $u_t(0, t)$.

Additional Problem 2 Find the solution to the wave equation $u_{tt} - 4u_{xx} = 0$ and with initial conditions $u(x, 0) = \sin x$, $u_t(x, 0) = 10$. Calculate then $u_t(0, t)$.

In the following 3 additional problems check first the second order PDE is hyperbolic

Additional Problem 3: Find the general solution to $u_{xx} + u_{xt} - 10u_{tt} = 0$ (check whether is hyperbolic first).

Additional Problem 4: Find the general solution to $u_{xx} + 2u_{xt} - 20u_{tt} = 0$ (check whether is hyperbolic first).

Additional Problem 5: Find the solution the IVP

$$\begin{cases} u_{xx} - 6u_{xt} + 5u_{tt} = 0, \qquad x \in \mathbb{R},\, t > 0 \\ u(x, 0) = x^2 \\ u_t(x, 0) = 0 \end{cases}$$

Check first whether the second order PDE is hyperbolic.

Bonus Problem:* Read Example 2 in Strauss' book pages 36-37. Then do Problem 5 Section 2.1 (do not turn in).

• Set 5. Due date: Monday March 30, 2020 (upload in Moodle as Single PDF document)

Additional Problem 1 Consider the wave equation in 1D with *damping*

$$u_{tt} = c^2\, u_{xx} - ku - ru_t \qquad k, r > 0$$

show that the *energy functional*

$$E(t) \;=\; \frac{1}{2} \int_{-\infty}^{\infty} |u_t|^2 + c^2 |u_x|^2 + k|u|^2 \, dx$$

satisfies $dE/dt \leq 0$; that is *energy decreases.* Assume u and its derivatives vanish as $x \to \pm\infty$.

Section 2.2: 1, 2, 3

Additional Problem 2: Let $u = u(\mathbf{x}, t)$ be a solution to the wave equation $u_{tt} - \Delta u = 0$ in $\mathbb{R}^2$. Assuming that $\nabla u \to 0$ fast enough as $|\mathbf{x}| \to \infty$ prove that

$$E(t) \;=\; \int_{\mathbb{R}^2} |u_t|^2 \;+\; |\nabla\, u|^2 \; dxdy$$

is constant in t for all time t.

Section 2.3: 2, 4, 5, 7a)*.

Hint for 4b): Do not solve explicitly. Rather prove that $u(1-x,t)$ also solves the equation and then apply the uniqueness theorem.

• Set 6. Due date: Thursday 04/9/2020

Section 2.4: 3, 4, 5a)b)c), 9, 11a), 15, 16.

Additional Problem a) Show that the function $u(x,t) = e^{-kt}\sin(x)$ solves the heat equation $u_t - ku_{xx} = 0$.

b) Find a relationship between the constants a and b so that $u(x,t) = e^{-at}\cos(bx)$ is a solution to $u_t - ku_{xx} = 0$ (assume $\cos(bx) \neq 0$).

• Set 7. Due date: Thursday 04/23/2020

Section 4.1: 2, 3.

Hint In 3) proceed as in the heat equation and keep the complex i next to $T(t)$. Recall that the solution to the ODE $T'(t) = i\lambda T(t)$ is $T(t) = Ae^{-i\lambda t}$.

Additional Problems (section 4.1) Separate variables to solve the following problems.

A1. $u_{tt} - u_{xx} = 0$ in $0 < x < 3$ with boundary conditions $u(0,t) = u(3,t) = 0$

A2. $u_t = u_{xx}$ in $0 < x < \pi$ with boundary conditions $u(0,t) = u(\pi,t) = 0$

A3. Let g be a smooth function on $[0,1]$, $g(0) = g(1) = 0$. Consider the Dirichlet BIVP:

$u_t - u_{xx} + 3u = 0$ in $0 < x < 1$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = g(x)$ $(\oplus)$

a) Consider the change of variables $u(x,t) = e^{-3t}v(x,t)$ and prove that v solves

$v_t - v_{xx} = 0$ in $0 < x < 1$ with $v(0,t) = v(1,t) = 0$ and $v(x,0) = g(x)$. $(\dagger)$

b) Use the method of separation of variables to find v, the solution to $(\dagger)$.

c) Use a) and b) to find u, the solution to $(\oplus)$.

Section 4.2:

Additional Problem 1 Separate variables to solve the following problem for the *wave equation* with zero Neumann boundary conditions, $u_{tt} - 4u_{xx} = 0$ in $0 < x < L$ and $u_x(0,t) = u_x(L,t) = 0$

Additional Problem 2 Separate variables to solve the following problem for the heat equation with zero Neumann boundary conditions, $u_t - 3u_{xx} = 0$ in $0 < x < \pi$ and $u_x(0,t) = u_x(\pi,t) = 0$

Read first the last part of section 4.2 (page 91) where there is an example with mixed boundary conditions. Then do: 1, 2 in Section 4.2.

• Set 8. Due date: Wednesday 05/06/2020

Section 5.1 2, 9

Hint for 9: Use the trig. identities $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1$ to re-express the initial velocity and immediately obtain its cosine "expansion".

Section 5.3 $3^\dagger$

Problem $3^\dagger$ means that you should do this problem for **zero Neumann boundary conditions instead of the mixed ones**. That is, consider the given wave equations with $u_x(0,t) = 0 = u_x(\ell,t)$ and the same initial data $u(x,0) = x$ and $u_t(x,0) = 0$.

Section 5.4 5, 8a)

Additional Problem* (do but do not turn in) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a periodic function with period p; that is $\phi(x+p) = \phi(x)$, $\forall x \in \mathbb{R}$. Assume that ϕ is integrable on any finite interval.

(a) Prove that for any $a, b \in \mathbb{R}$

$$\int_a^b \phi(x)dx = \int_{a+p}^{b+p} \phi(x)dx = \int_{a-p}^{b-p} \phi(x)dx$$

(b) Prove that for any $a \in \mathbb{R}$

$$\int_{-p/2}^{p/2} \phi(x+a)dx = \int_{-p/2+a}^{p/2+a} \phi(x)dx = \int_{-p/2}^{p/2} \phi(x)dx$$

Note then that for any $a \in \mathbb{R}$, $\int_{-p/2}^{p/2} \phi(x+a)dx = \int_{-p/2}^{p/2} \phi(x)dx$ and thus in particular that $\int_a^{a+p} \phi(x)\,dx$ does not depend on a, as we discussed in class (section 5.2, Strauss).

Special Assignments

Please do the following but do not turn in yet. The solutions to these special projects must be typed using Latex. Due date: TBA

Special Project 1

Consider the *initial value problem for the wave equation* on $\mathbb{R}$:

$$\begin{cases} u_{tt} - u_{xx} &= 0 \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{cases}$$

where $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are two smooth given functions (data). Let $x_0 \in \mathbb{R}$ $t_0 > 0$ be fixed and suppose that $\phi(x)$ and $\psi(x)$ vanish for all x in the interval $[x_0 - t_0, x_0 + t_0]$.

Finite Propagation Speed Theorem:. The solution $u(x,t)$ to the initial value problem above vanishes for all (x,t) within $\mathcal{C}$, the domain of dependence of (x_0, t_0). Recall

$$\mathcal{C} := \{(x,t) : 0 \leq t \leq t_0 \text{ and } x_0 - (t_0 - t) \leq x \leq x_0 + (t_0 - t)\}.$$

Remark: The Theorem is also valid in higher dimensions but for simplicity I will ask you to prove it only in one (space) dimension. In one dimension, one can trivially prove the above theorem directly using the representation formulas for the solution $u(x,t)$ in terms of the initial data which are available in one dimension. Or, one could prove it without using this explicit representation of u, but by using the *energy method* instead –as we have seen in class-. This proof is a bit harder but the advantage of the method is that it also works in higher dimensions.

The project consists then to prove the Finite Propagation Speed Theorem above using the energy method.

To do so, for each $0 \leq t \leq t_0$, let $I_t := [x_0 - (t_0 - t), x_0 + (t_0 - t)]$.
Note I_t is contained in the interval $(x_0 - t_0, x_0 + t_0)$. Define the modified energy:

$$\widetilde{E}(t) = \frac{1}{2} \int_{I_t} |u_t|^2 + |u_x|^2 \, dx$$

Note $\widetilde{E}(t) \geq 0$ for any t and that $\mathcal{C} = \bigcup_{0 \leq t \leq t_0} I_t$. The goal is to show that for each $0 \leq t \leq t_0$, $u(x,t) = 0$ for all $x \in I_t$. Do so by proving the following:

(1) Prove that $\widetilde{E}(t)$ is a decreasing function of t by showing that $\dfrac{d\widetilde{E}}{dt} \leq 0$

To compute the derivative in time use: (see A.3 Theorem 3 in Strauss's book p.421).

$$\frac{d}{dt}\int_{a(t)}^{b(t)} F(x,t)\,dx = \int_{a(t)}^{b(t)} \frac{d}{dt}F(x,t)\,dx + [F(b(t),t)b'(t) - F(a(t),t)a'(t)]$$

(2) Show that $\widetilde{E}(0) = 0$

(3) By (1) you then have that $\widetilde{E}(t) \leq \widetilde{E}(0)$ for any $0 \leq t \leq t_0$ and by (1) you can conclude that $\widetilde{E}(t) = 0$ for any $0 \leq t \leq t_0$. Prove then that this implies that $u(x,t) = 0$ for any $x \in I_t$ and any $0 \leq t \leq t_0$.

Special Project 2 (a) Solve the following hyperbolic initial value problem on $\mathbb{R}$ by first completing the square and solve the equation in terms of generic functions f and g. Then use the initial conditions to choose appropriate f, g and constants.

$$\begin{cases} u_{xx} + 2u_{xt} - 80u_{tt} = 0 \\ u(x,0) = x^2 \\ u_t(x,0) = 0 \end{cases}$$

(b) Consider the inhomogeneous problem for the wave equation on $[0, L]$:

$$\text{(WE)} \qquad \begin{cases} u_{tt} - u_{xx} = f(x,t) & t > 0 \\ u(0,t) = g(t), \quad u(L,t) = h(t) \\ u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x) \end{cases}$$

(i) Are the boundary conditions of (WE) of Dirichlet or Neumann type?

(ii) Prove the uniqueness of solutions to this problem using the energy method.
Hint. Consider the difference w of two possible solutions u_1 and u_2 to (WE) and use the energy conservation of energy applied to w.

Special Project 3

(a) Consider now the initial value problem for the diffusion equation on the **whole** real line $\mathbb{R}$ with $k = 1$:

$$(*) \qquad \begin{cases} u_t - u_{xx} = 0, & x \in \mathbb{R}, \ t > 0 \\ u(x,0) = e^{2x}, & x \in \mathbb{R} \end{cases}$$

Use the fact that the solution $u(x,t)$ is obtain by the convolution of the fundamental solution with the initial data; that is by:

$$u(x,t) \;=\; \int_{-\infty}^{\infty} \Gamma_k(x-y,t)\, e^{2y}\, dy$$

to find the function that $u(x,t)$ equals to. Check that your answer solves indeed $(*)$.

To solve proceed as follows:

1) Recall that in 1D, $\Gamma_k(x-y,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}}$, $t>0$ (in Strauss notation this is $S(x-y,t)$). Note that here we have $k=1$.

2) After developing the square in Γ, collect all the exponents of the exponentials and **complete the square in the** y variable. Note that terms that have only x and t in the exponents can come out of the integral.

3) You may use that $\int_{-\infty}^{\infty} e^{-p^2}\, dp = \sqrt{\pi}$. You may find the change of variables $p = \frac{y-(x+4t)}{\sqrt{4t}}$ useful.

(b) Let $u_1(x,t)$ and $u_2(x,t)$ be solutions to the heat equation $u_t = k\, u_{xx}$, with initial and boundary conditions: $u_1(x,0) = f_1(x),\ u_1(0,t) = g_1(x),\ u_1(L,t) = h_1(t),$ and $u_2(x,0) = f_2(x),\ u_2(0,t) = g_2(x),\ u_2(L,t) = h_2(t)$ respectively.

Assume that $f_1 \geq f_2$, $g_1 \geq g_2$ and $h_1 \geq h_2$. Prove that then $u_1 \geq u_2$ in the region $\mathcal{R} = [0,L]\times[0,\infty)$.

Hint. Consider $w = u_1 - u_2$, set up an appropriate boundary-initial value problem for w and use the max or the min principle (specify) to prove that $w \geq 0$ on $\mathcal{R}$.

Special Project 4

a) [Wave on the half line. **Use Handout 9)**]

Find the solution to the following wave equation on the half-line using the reflection method. Show all your work.

$$\begin{cases} u_{tt} - 4u_{xx} \;=\; 0 \\ u(x,0) = 1, \quad u_t(x,0) = 0 \\ u(0,t) = 0 \end{cases}$$

The solution has a jump discontinuity in the (x,t) plane. Find its location (explain).

b) [Wave with a source. **Use Handout 10)**].

Find the solution to the following inhomogeneous wave equation on $\mathbb{R}$. Evaluate all the integrals to obtain a nice formula for the solution

$$u_{tt} - 9u_{xx} = x\,t \qquad u(x,0) = \sin(x) \qquad u_t(x,0) = 1 + x$$

Final Problem 5

a) Find the Fourier cosine series of $\phi(x) = x^2$ for $x \in [0,1]$

b) State in what sense does the cosine series in part a) converges to the function x^2 on $[0,1]$.

c) Use separation of variables and the superposition principle to find the general solution to the following boundary value problem for the heat equation on an interval:

$$(H) \qquad \begin{cases} u_t - u_{xx} = 0, & 0 < x < 1, \ t > 0 \\ u_x(0,t) = 0 = u_x(1,t) & t > 0 \end{cases}$$

In the course of your proof do an analysis of all the possible eigenvalues ($\lambda > 0$, $\lambda = 0$, $\lambda < 0$) to the problem,

$$\begin{cases} X'' + \lambda X(x) = 0 & 0 < x < 1 \\ X'(0) = 0 = X'(1) \end{cases}$$

d) Find the particular solution to (H) that also satisfies the initial condition that $u(x,0) = x^2$, for $0 < x < 1$.