The Heat Equation on $\mathbb{R}^d$, $d \geq 1$

Now there is no (finite) boundary conditions to worry about. Our problem looks like

$$(\dagger)\begin{cases} u_t - k\Delta u = f(x,t); & x \in \mathbb{R}^d,\ t>0 \\ u(x,0) = g(x); & x \in \mathbb{R}^d \end{cases}$$

or in 1D simply:

$$(\dagger)'\begin{cases} u_t - k u_{xx} = f(x,t) & x \in \mathbb{R};\ t>0 \\ u(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

Recall that if $f(x,t) \equiv 0$ for all x, t then the equation is (linear) homogeneous.

Remark: We'll see that we have to impose some conditions for the date/solution at spatial infinity (that is as $|x| \to \infty$, "boundary at ∞").

We are able to represent solutions to $(\dagger)$ (or $(\dagger)'$) in terms of f and g thanks to a very special solution to the homogeneous eq. called the FUNDAMENTAL SOLUTION :

DEFINITION: The fundamental solution $\Gamma_k(x,t)$ is defined to be

$$\Gamma_k(x,t) = \frac{1}{(4\pi k t)^{d/2}} e^{-|x|^2/4kt} \quad \text{for } t>0,\ x \in \mathbb{R}^d$$

Recall $x = (x_1, x_2, \dots, x_d)$

$|x|^2 = x_1^2 + x_2^2 + \dots + x_d^2$

Lemma: $\Gamma_k(x,t)$ solves $u_t - k\Delta u = 0$ $x \in \mathbb{R}^d$, $t>0$.

Proof: We need to show that

$$(\Gamma_k)_t = k\Delta(\Gamma_k)$$

$$\partial_t \Gamma_k(x,t) = \left(\frac{-2\pi k d}{(4\pi k t)^{d/2+1}} + \frac{1}{(4\pi k t)^{d/2}} \frac{|x|^2}{4kt^2} \right) e^{-|x|^2/4kt}$$

$$\partial_{x_i} \Gamma_k(x,t) = -\frac{2\pi x_i}{(4\pi k t)^{d/2+1}} e^{-|x|^2/4kt}$$

$$\partial^2_{x_i} \Gamma_k(x,t) = \left(\frac{-2\pi}{(4\pi k t)^{d/2+1}} + \frac{1}{(4kt)} \frac{4\pi x_i^2}{(4\pi k t)^{d/2+1}} \right) e^{-|x|^2/4kt}$$

So

$$k\Delta \Gamma_k(x,t) = \left(\frac{-2\pi k d}{(4\pi k t)^{d/2+1}} + \frac{1}{\cancel{4}\cancel{k}t} \frac{\cancel{4}\pi \cancel{k} |x|^2}{(4\pi k t)^{d/2+1}} \right) e^{-|x|^2/4kt}$$

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PROPERTIES : 1 For $x \neq 0$ $\lim_{t \to 0^+} \Gamma_k(x,t) = 0$

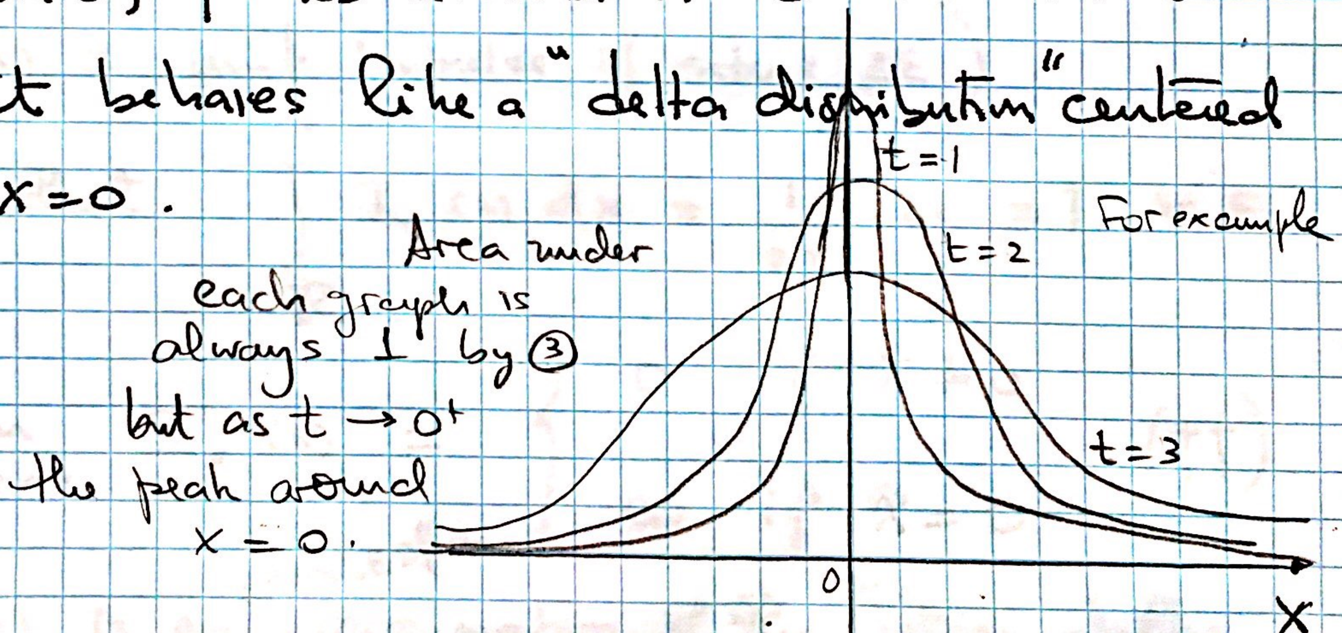
2 At $x=0$ $\lim_{t \to 0^+} \Gamma_k(0,t) = \infty$

$$\shortparallel$$
$$\frac{1}{(4\pi k t)^{d/2}}$$

3 $\displaystyle\int_{\mathbb{R}^d} \Gamma_k(x,t)\, dx = 1 \quad \forall t>0$ (Areas always 1

(recall $dx = dx_1 dx_2 \dots dx_d$).

Remark : $\Gamma_k(x,t)$ is a gaussian-type function
The properties above suggest that as $t \to 0^+$
$\Gamma_k(x,t)$ peaks around $x=0$ and in the limit behaves like a "delta distribution" centered at $x=0$.



Remark : Property 3 follows from a change of variables and the fact that $\displaystyle\int_{\mathbb{R}^d} e^{-|z|^2} dz = \pi^{d/2}$.

Discussion about the delta distribution (also called Dirac delta) centered at 0. on $\mathbb{R}$

Consider the Heaviside function

$$H(x) := \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad x \in \mathbb{R}$$

and let

$$I_\varepsilon(x) := \frac{H(x+\varepsilon) - H(x-\varepsilon)}{2\varepsilon}$$

then

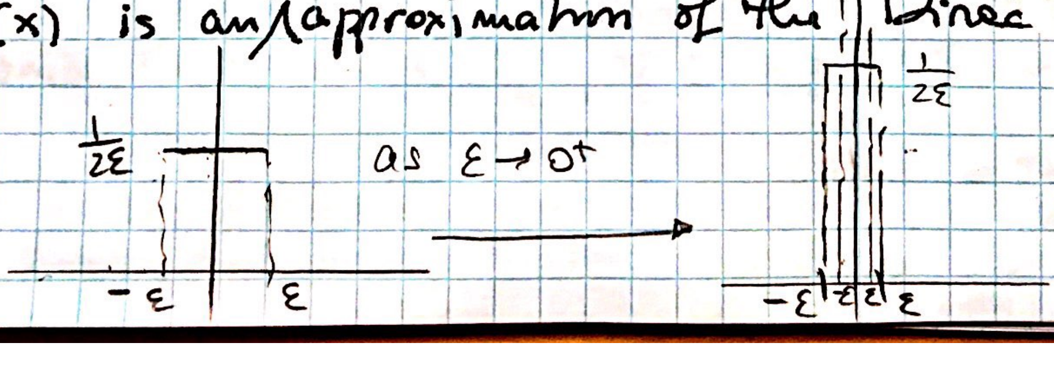
$$I_\varepsilon(x) = \begin{cases} \frac{1}{2\varepsilon} & -\varepsilon \leq x \leq \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

($I_\varepsilon(x)$ = unit impulse of extent 2ε)

Note that $\displaystyle\int_{\mathbb{R}} I_\varepsilon(x)\,dx = \frac{1}{2\varepsilon}\cdot 2\varepsilon = 1 \quad \forall \varepsilon$

$$\lim_{\varepsilon \to 0^+} I_\varepsilon(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \qquad (\dagger\dagger)$$

$I_\varepsilon(x)$ is an EXAMPLE of an approximation of the Dirac delta



The Dirac δ satisfies as well that

- δ at $x=0$ is ∞ (" $\delta(0)=\infty$ " → NOTATION BUT δ IS NOT a FUNCTION).
- δ at $x\neq 0$ is 0
- " $\int_{\mathbb{R}} \delta(x)\,dx$ " $= 1$ meaning (††) for any approximation.

The $\Gamma_{\mathbb{R}}(x,t)$ is a smoother version of $I_\varepsilon(x)$ where instead of a step function we see a gaussian and the behavior as $\varepsilon\to 0^+$ is seen as $t\to 0^+$.

To rigorously understand how $I_\varepsilon \to \delta$ as $\varepsilon\to 0^+$ (similar idea works for $\Gamma_h(x,t)$ as $t\to 0^+$)

we consider a smooth function φ with compact support or decaying very fast at infinity. Then

$$\int_{-\infty}^{\infty} I_\varepsilon(x)\,\varphi(x)\,dx = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\varphi(x)\,dx \xrightarrow[\substack{\varepsilon\to 0^+\\ \text{FTC}}]{} \varphi(0)$$

We write this as $\langle I_\varepsilon,\varphi\rangle \xrightarrow[\varepsilon\to 0]{} \langle \delta,\varphi\rangle := \varphi(0)$

THIS IS THE DEFINITION OF δ

Remark: In terms of H itself we're saying that " $H'=\delta$ " $\int H'\varphi = -\int_0^\infty H\varphi' = -\int_0^\infty \varphi' = \varphi(0)$

So formally then the definition of Dirac' δ is

<u>DEF</u>: The Dirac δ distribution (centered at 0) is a "generalized function" (meaning distribution) that <u>acts</u> <u>on</u> (nice) <u>functions</u> $\varphi(x)$ as follows

$$\langle \delta, \varphi \rangle := \varphi(0)$$

($\langle \, , \rangle$ is notation meaning δ acting on φ (or pairing of δ with φ)).

<u>Remark</u>. We can shift the center of the Dirac δ so that it peaks at some other x than the origin. We write this as follows

$$\langle \delta(x - x_0), \varphi \rangle := \varphi(x_0). \quad x_0 \text{ fixed}$$

Note for example $I_\varepsilon(x - x_0) \xrightarrow[\varepsilon \to 0^+]{} \delta(x - x_0)$

and similarly $\Gamma_k(x - x_0, t) \xrightarrow[t \to 0^+]{} \delta(x - x_0)$

We formalize the statements above for Γ_k in the following <u>Lemma</u>, <u>Remarks</u> and <u>Proposition</u>.

- In the Lemma, we'll allow the smooth function φ to grow a bit at ∞ (doesn't have to!) b/c the $\Gamma_k(x,t)$ is gaussian which means decays very fast - exponentially fast - and can "absorb" some growth when being integrated.

Lemma: Suppose that φ is a smooth function on $\mathbb{R}^d$ and that $\exists$ constants $a, b \geq 0$ such that

$$|\varphi(x)| \leq a e^{b|x|^2}$$

Then

$$\lim_{t \to 0^+} \int_{\mathbb{R}^d} \Gamma_k(x,t)\, \varphi(x)\, dx = \varphi(0)$$

Remark. We can restate the Lemma above as

$$\lim_{t \to 0^+} \langle \Gamma(\cdot, t), \varphi(\cdot) \rangle = \langle \delta, \varphi \rangle$$
$$= \varphi(0).$$

Or simply as

$$\lim_{t \to 0^+} \Gamma_k(x,t) = \delta(x) \quad \left(\text{"} \Gamma_k(x,0) = \delta(x) \text{"} \right)$$

PROPOSITION: $\Gamma_k(x,t)$ is a solution to $u_t - k\Delta u = 0$ verifying the initial conditions

$$\lim_{t \to 0^+} \Gamma_k(x,t) = \delta(x).$$

Convolutions (Aside; we need this for what follows)

If f and g are two functions on $\mathbb{R}^d$ then the convolution of f and g, denoted by $f * g$ is a new function on $\mathbb{R}^d$ defined as:

$$f * g(x) = \int_{\mathbb{R}^d} f(y)\, g(x-y)\, dy$$

Provided f, g are nice so all integrals make sense.

$$= \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$$

$$= g * f(x)$$

Roughly speaking $f * g$ can be viewed as an "averaging" of g relative to f.

We can extend convolutions to the δ Dirac distribution as follows:

$$(\delta * g)(x) = \langle \delta(x-y), g(y) \rangle = g(x)$$

$\langle \delta(x-y), g(y) \rangle \to$ pairing in y for x fixed

Note $\langle \delta(x-y), g(y) \rangle = \langle \delta(y), g(x-y) \rangle = g(x-0) = g(x)$.

The Cauchy Problem for the diffusion/heat eq. on $\mathbb{R}^d$

We are now ready to find a representation formula for solutions to the Cauchy IVP on $\mathbb{R}^d$:

$$(\dagger\dagger)\quad \begin{cases} u_t - k\,\Delta u = 0 & x \in \mathbb{R}^d,\ t>0 \\ u(x,0) = g(x) & x \in \mathbb{R}^d \end{cases} \qquad \boxed{\text{HOMOGENEOUS CASE}}\ .$$

THEOREM

Assume that $g : \mathbb{R}^d \to \mathbb{R}$ is a continuous (or smooth) function such that at most g grows at ∞ no faster than $a e^{b|x|^2}$ $a, b > 0$

That is $|g(x)| \le a e^{b|x|^2}$ (it may not grow at all but stay bounded or grow for ex. polynomially which is slower than $a e^{b|x|^2}$ for some $a, b > 0$)

Then, there exists a solution $u(x,t)$ to the homogeneous eq. $(\dagger\dagger)$ for $(x,t) \in \mathbb{R}^d \times [0, T)$ where

$$T := \frac{1}{4kb} \quad \left(\text{time of existence}\right)$$

Furthermore, $u(x,t)$ can be represented as

$$\left.\begin{aligned} u(x,t) &= \Gamma_k(\cdot, t) * g\,(x) \\ &= \int_{\mathbb{R}^d} \Gamma_k(x-y, t)\, g(y)\, dy\ . \end{aligned}\right] \text{(Sol.)}$$

The solution $u(x,t)$ is of regularity C^∞ on $\mathbb{R}^d \times [0,T)$ (even if g is just continuous, u is C^∞)

Finally, for each compact subinterval $[0,T'] \subset [0,T)$ $\exists\, A, B > 0$ (dep. on the compact subinterval) s.t

$$(*) \qquad |u(x,t)| \leq A\, e^{B|x|^2}$$

for all $(x,t) \in \mathbb{R}^d \times [0,T']$. The solution $u(x,t)$ is the UNIQUE solution in the class of functions verifying $(*)$

<u>Remarks:</u> (i) Note that $T := \dfrac{1}{4kb}$ goes to ∞ as $b \to 0$. So if g is bounded the time T of existence is infinity ie. the solution exists on $\mathbb{R}^d \times [0,\infty)$.

(ii) Also note that the representation formula (Sol.) shows that the solution to the IVP propagates with <u>INFINITE SPEED</u> : the value of g at a point $x_0 \in \mathbb{R}^d$ has an immediate effect everywhere for u. More precisely even if g has compact support (Sol) shows that at any $t>0$, the solution $u(x,t)$ has spread out over the entire space $\mathbb{R}^d$.

(iii) To see that u as in (Sol.) solves the equation note that if $\mathcal{L} := \partial_t - k\Delta_x$ then

$$\mathcal{L}u(x,t) \underset{\substack{\text{assuming we}\\ \text{justify differentiation}\\ \text{under the integral sign}}}{=} \int_{\mathbb{R}^d} \underbrace{\mathcal{L}\,\Gamma_k(x-y,t)}_{=0}\, g(y)\,dy = 0 \quad \begin{matrix}(x\in\mathbb{R}^d)\\ (t>0)\end{matrix}$$

differentiates in t and in x so it falls all in $\Gamma_k(x-y,t)$.

and
$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} \left(\Gamma_k(\cdot,t) * g\right)(x)$$
$$= \langle \delta(x-y), g(y)\rangle = \delta * g\,(x)$$
$$= g(x).$$

- Next we would like to know how to modify (Sol.) to treat the <u>INHOMOGENEOUS equation</u>

$$(\dagger\dagger\dagger)\begin{cases} u_t - k\Delta u = f(x,t) & x\in\mathbb{R}^d,\ t>0 \\ u(x,0) = g(x), & x\in\mathbb{R}^d \end{cases}$$

The answer is given by the <u>DUHAMEL PRINCIPLE</u> encoded in the following theorem.

We assume that g is as above and $T = \frac{1}{4kb}$ as well as above. We also assume that,

$f(x,t)$, $\partial_{x_i} f(x,t)$ and $\partial_{x_i}\partial_{x_j} f(x,t)$ are continuous and bounded on $\mathbb{R}^d \times [0,T)$ for all $1 \le i, j \le d$.

THEOREM: There exists a unique solution $u(x,t)$ to the INHOMOGENEOUS IVP (†††) on $\mathbb{R}^d \times [0,T)$. Furthermore, $u(x,t)$ can be represented as

$$u(x,t) = \underbrace{\left(\Gamma_k(\cdot, t) * g\right)(x)}_{\text{Solution to HOMOGENEOUS equation as in (Sol.) with data } g} +$$

$$+ \underbrace{\int_0^t \left(\Gamma_k(\cdot, t-s) * f(s, \cdot)\right)(x)\, ds}_{\text{solution to inhomog. eq. with 0 data.}}$$

The solution $u \in C^{2,1}(\mathbb{R}^d \times (0,T)) \cap C^0(\mathbb{R}^d \times [0,T))$

Remarks (i) In other words $u(x,t)$ is the sum of the solution to the homogeneous problem

$$\begin{cases} \partial_t u - k\Delta u = 0 \\ u(x,0) = g \end{cases}$$

+ sol. to inhomogeneous equation

$$\begin{cases} \partial_t u - k\Delta u = f \\ u(x,0) = 0 \end{cases}$$

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(ii) The term $\displaystyle\int_0^t \left(\Gamma_k(\cdot, t-s) * f(s,\cdot)\right)(x)\, ds$

$$\overset{\text{is}}{=} \int_0^t \left[\int_{\mathbb{R}^d} \Gamma_k(x-y, t-s)\, f(s,y)\, dy\right] ds$$

To see that this term satisfies

that $\mathcal{L}$ of it equals f we need to differentiate it w.r.t to t and since t appears both in the integral and the integrand to compute $\partial_t \left(\int_0^t \cdots\right)$ we need to use Problem 8 Set 4.

The Δ_x passes through the integrals.

Note that at $t=0$ this term is 0 (b/c $\int_0^t$) $\overset{t=0}{\downarrow}$

Lastly we briefly go through the derivation of $\Gamma_k(x,t)$. This relates to Problem 7 Set 4.

Invariances: (1) If u solves $\partial_t u - k\Delta u = 0$ then for $A, t_0 \in \mathbb{R}$ constants and $x_0 \in \mathbb{R}^d$ fixed then $u^*(x,t) := A\, u(x-x_0, t-t_0)$ also satisfies $u^*_t - k\Delta u^* = 0$.

2. If $\lambda > 0$ then $u_\lambda(x,t) := A\,u(\lambda x, \lambda^2 t)$

is also a solution, ie $(u_\lambda)_t - k\Delta(u_\lambda) = 0$.

3. Suppose that $Z(t) := \int_{\mathbb{R}^d} u(x,t)\,dx$

This is called the <u>total thermal energy</u> and one can show that (for rapidly decaying solutions as $|x| \to \infty$) $Z(t)$ is constant in time, that is $\frac{d}{dt} Z(t) = 0$.

So $Z(t) = Z(0)$.

Then if one chooses $A = \lambda^d$ in 2 the

solution $u^*_\lambda(x,t) = \lambda^d u(\lambda x, \lambda^2 t)$

satisfies that $\int_{\mathbb{R}^d} u^*_\lambda(x,t)\,dx = Z(t)$

$$= \int_{\mathbb{R}^d} u(x,t)\,dx.$$

That is u^*_λ conserves the total thermal energy of u.

To find the fundamental solution given here in page 2 we consider the parabolic scaling and define

$$z = \frac{x}{\sqrt{kt}}$$

Then z is invariant under $t \to \lambda^2 t$
$x \to \lambda x$

that is $z = \dfrac{x}{\sqrt{kt}} = \dfrac{\lambda x}{\sqrt{k\lambda^2 t}}$

We go through the derivation of $\Gamma_k(x,t)$ in 1D, that is when $\underline{x \in \mathbb{R}}$, $t>0$.

<u>Ansatz</u>: Look for solutions to $\partial_t - k\partial_{xx} = 0$ of the form

$$\frac{1}{\sqrt{kt}} V(z) = \frac{1}{\sqrt{kt}} V\left(\frac{x}{\sqrt{kt}}\right)$$

need to determine V

(this will be our $\Gamma_k(x,t)$)

We want $\displaystyle 1 = \int_{\mathbb{R}} \Gamma_k(x,t)\,dx \Rightarrow$

we need $\displaystyle 1 = \int_{\mathbb{R}} V(z)\,dz$ since $dz = \frac{1}{\sqrt{kt}}\,dx$

If WE WANT THAT $(\partial_t - k\partial_{xx})(\Gamma_k(x,t)) = 0$ then we need

that

(*) $$V''(z) + \frac{1}{2} z V'(z) + \frac{1}{2} V(z) = 0$$

- We impose/demand that $\boxed{V(z) \geq 0}$ and
- that $\boxed{\text{as } z \to \pm\infty \quad V(z) \to 0}$

[This is b/c we expect Γ_k to behave like δ for small $t>0$ and also that $\Gamma_k(x,t)$ decays rapidly as $|x| \to \infty$]

- We also want V to be even $V(z) = V(-z)$

(b/c we want both $V(z)$ and $V(-z)$ to solve (*)

Hence it follows that $\boxed{V'(0) = 0}$

Then we can rewrite (*) as

$$\frac{d}{dz}\left(V'(z) + \frac{1}{2} z V(z)\right) = 0$$

$$\Rightarrow \quad V'(z) + \frac{1}{2} z V(z) = \underline{\text{constant}}$$

By setting $z=0$ and using that $V'(0)=0$ we see that then, this <u>constant</u> must be 0. That is we now have that

$$V'(z) + \frac{1}{2} z V(z) = 0.$$

$\Longrightarrow$ Solving the ODE

$$\ln\left(\frac{V(z)}{V(0)}\right) = -\frac{1}{4} z^2$$

$$\Rightarrow \quad V(z) = V(0)\, e^{-\frac{1}{4} z^2}$$

Since $\int_{\mathbb{R}} V(z)\, dz = 1 \Rightarrow V(0) = \frac{1}{\sqrt{4\pi}}$

(use that $\int_{\mathbb{R}} e^{-y^2} dy = \sqrt{\pi}$).

$$\therefore \quad V(z) = \frac{1}{\sqrt{4\pi}} e^{-\frac{1}{4} z^2}$$

$$\therefore \quad \Gamma_k(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/4kt}$$

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