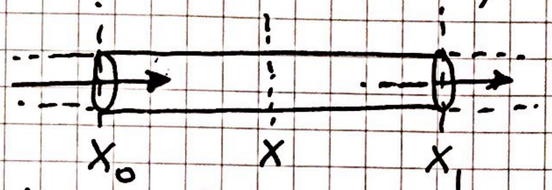
DIFFUSION AND HEAT FLOW (from W. Strauss book on PDE.)

1) Diffusion: imagine a motionless liquid filling a straight pipe



and a chemical substance (such as a dye) diffusing through the liquid. Simple diffusion is characterized by Fick's law of diffusion: the dye moves from regions of higher concentration to regions of lower concentration. The rate of motion is PROPORTIONAL to the concentration gradient.

Let $u(x,t)$ = concentration (mass per unit length) of the dye at position x of the pipe at time t.

In the section of the pipe from -say- x_0 to x_1 (as in figure above), the mass of dye is

$$M(t) = \int_{x_0}^{x_1} u(x,t)\,dx \quad \text{so} \quad \frac{dM}{dt} = \int_{x_0}^{x_1} \underbrace{\frac{du}{dt}(x,t)}_{=u_t(x,t)}dx$$

The mass in this section of the pipe cannot change except by flowing in or out of its ends.

By Fick's law: $\frac{dM}{dt}$ = flow in − flow out

$$= k u_x(x_1,t) - k u_x(x_0,t)$$

where k is a proportionality constant.

$(k > 0)$

(bcam) basque center for applied mathematics

Therefore those 2 expressions are equal:

$$\int_{x_0}^{x_1} u_t(x,t)\,dx = k u_x(x_1,t) - k u_x(x_0,t)$$

Differentiating w.r.t. x_1 we get $\boxed{u_t = k\, u_{xx}}$

Diffusion equation in 1D.

• In 3D we have instead:

$$\iiint_D u_t(x,y,z,t)\,dx\,dy\,dz = \iint_{\partial D} k(\vec{n}\cdot\nabla u)\,dS$$

where D is any solid and ∂D = boundary surface.

By the divergence theorem we get the 3D-diffusion eq.

$$u_t = k(u_{xx} + u_{yy} + u_{zz})$$

that is $\quad u_t = k\,\Delta u$

If there is an external source (or "sink") of the dye.

The same equation describes the CONDUCTION OF HEAT (as well as many other phenomena such as population dynamics, Brownian motion, etc.)

2) <u>HEAT FLOW</u> (on $D \subseteq \mathbb{R}^3$; $(x,y,z) \in D$)

Let $u(x,y,z,t)$ be the temperature and let $H(t)$ be the amount of heat (say in calories) contained in a region $D \subseteq \mathbb{R}^3$. Then

$$H(t) = \iiint_D c\rho\, u(x,y,z,t)\, dx\,dy\,dz \,,$$

where c is the "specific heat" (constant) of the material and ρ is its density (mass per unit volume)

($c, \rho > 0$ constants.)

The change in heat is

$$\frac{dH}{dt} = \iiint_D c\rho\, u_t(x,y,z,t)\, dx\,dy\,dz \,.$$

FOURIER'S LAW says that heat flows <u>from</u> HOT <u>TO</u> cold regions proportionately to the temperature gradient. But heat cannot be lost from D except by leaving it through the boundary. This is the <u>LAW of CONSERVATION of energy</u>. Therefore, the change of heat energy in D also equals the heat flux across the boundary,

$$\frac{dH}{dt} = \iint_{\partial D} k\,(\vec{n} \cdot \nabla u)\, d\sigma$$

$\vec{n}$: normal to the boundary ∂D

$d\sigma$: surface measure

[bcam] basque center for applied mathematics

where $K > 0$ is a proportionality factor (the "heat conductivity"),

By the divergence theorem,

$$\iiint_D c\rho \frac{du}{dt}\, dxdydz = \iiint_D \nabla\cdot(K\nabla u)\, dxdydz$$

(and the above holds for all D)

Since $K>0$ is constant, $\nabla\cdot(K\nabla u) = K\Delta u$;

so we get the heat equation

$$c\rho \frac{\partial u}{\partial t} = K\Delta u$$

or

$$\frac{\partial u}{\partial t} = k\Delta u \qquad \left(k = \frac{K}{c\rho} > 0\right)$$

(HOMOGENEOUS equation)

Heat FLOW (REVISITED in language of Salsa's book)

Consider a homogeneous, isotropic solid body $B \subset \mathbb{R}^n$ ($n=3$ is most physically relevant case) described by the following physical properties:

ρ := mass density $\sim$ [mass] $\times$ [Volume]$^{-1}$ = constant

$e(t,x)$:= thermal energy per unit mass $\sim$ [energy] $\times$ [mass]$^{-1}$.

Let's also assume that heat is supplied to the body an external source which pumps in heat at the rate (per unit mass) $R = R(t,x)$, where

(bcam) basque center for applied mathematics

$$R \sim [\text{energy}] \times [\text{time}]^{-1} \times [\text{mass}]^{-1}$$

The TOTAL THERMAL energy $E(t; V)$ contained in a body sub-volume $V \subset B$ at time t is the integral of $e(t,x)$ over V:

$$E(t; V) := \int_V \rho\, e(t,x)\, dx \qquad x = (x_1 x_2 \dots x_n)$$

$$dx = dx_1 dx_2 \dots dx_n$$

The rate of change of the total energy contained in V is:

$$\frac{dE}{dt}(t; V) := \frac{d}{dt} \int_V \rho\, e(t,x)\, dx \qquad (\dagger)$$

(we assume e is "nice" so that we can differentiate under the integral) $\longrightarrow$

$$= \int_V \rho\, \partial_t e(t,x)\, dx$$

The rate of energy pumped into the sub-volume V by the external source:

$$(\dagger\dagger) \qquad \int_V \rho\, R(t,x)\, dx \sim [\text{energy}] \times [\text{time}]^{-1}$$

<u>We assume</u> also that heat energy is flowing throughout the body, and that the flow can be modeled by a <u>HEAT FLUX VECTOR</u> $\vec{q} \sim [\text{energy}] \times [\text{time}]^{-1} \times [\text{area}]^{-1}$, which specifies the direction and magnitude of heat flow across a unit area. That is if $d\sigma \subset \partial V$ is a small surface area with outward unit normal $\vec{n}$, then

(bcam) basque center for applied mathematics

$\vec{q} \cdot \vec{n}$ is the energy flowing out of the small surface. Thus, the rate of heat energy flowing into V is

$$-\int_{\partial V} \vec{q} \cdot \vec{n} \, d\sigma = -\int_V \nabla \cdot \vec{q} \, dx \sim [\text{energy}] \times [\text{time}]^{-1} \qquad (\dagger\dagger\dagger)$$

where the equality follows from the <u>divergence theorem</u>.

We also assume the following <u>ENERGY CONSERVATION LAW</u>:

The rate of change of total energy in the sub-volume V is equal to the rate of heat energy flowing into V + rate of heat energy supplied by the external source.

By ($\dagger$), ($\dagger\dagger$) and ($\dagger\dagger\dagger$) we see that this ENERGY CONSERVATION LAW takes the form:

$$\int_V \rho \, \partial_t e(t,x) \, dx = -\int_V \nabla \cdot \vec{q} \, dx + \int_V \rho R \, dx$$

Since this holds for all sub-volumes V, the integrands must be equal (assuming they are all "nice" - smooth -):

Then:

$$\rho \, \partial_t e(t,x) = -\nabla \cdot \vec{q} + \rho R$$

and by Fourier Law we have $\vec{q}(t,x) = -K \nabla u(t,x)$

(thermal conductivity, constant)

Finally, we assume that $e = c_v u$ ($c_v > 0$ is a specific heat at constant volume). (model).

All in all we then have

$$\partial_t u(t,x) = \frac{K}{c_v \rho} \Delta u + \frac{1}{c_v} R$$

$$\partial_t u(t,x) = k \Delta u + f(t,x)$$

$k = \dfrac{K}{c_v \rho}$ and $f(t,x) = \dfrac{1}{c_v} R(t,x)$.

#