Derivation of Wave Equation in 1 Dimension

- Vibrating String: flexible homogeneous string of length $L$
  
  ![Image of a string with displacement $u(x,t)$ at time $t$ fixed]

- Assume the string is vibrating on the $(x,u)$-plane. Then $u(x,t) =$ displacement from equilibrium (rest position) at time $t$ and position $x$.

Look at an infinitesimally piece of string (far from the endpoints). For example:

![Image of a displaced string segment]

- String is perfectly flexible $\Rightarrow$ tension (force) is directed tangentially along the string.

Let $T(x,t) =$ magnitude of this tension vector.

Let $\rho =$ density (mass per unit length) of string (string homogeneous $\Rightarrow \rho =$ constant).

Recall Newton's 2nd Law: $F = ma$.
On the other hand, that the projections of $T$ onto its vertical and horizontal components are respectively $T\sin \theta$ and $T\cos \theta$ and that ($\theta$ is angle as in drawing in page 11)

Then by Newton's second law we have for the horizontal and vertical forces that

\begin{align*}
(1) & \quad \frac{T \cos \theta}{\sqrt{1 + u_x^2}} \bigg|_{x = x_0}^{x = x_1} = 0 \quad \text{since there is no motion in the longitudinal direction,} \\
(2) & \quad \frac{T \sin \theta}{\sqrt{1 + u_x^2}} \bigg|_{x = x_0}^{x = x_1} = \int_{x_0}^{x_1} \rho \frac{u}{u_t} \text{ vertical acceleration}
\end{align*}

Sum of vertical forces at $x = x_0$ and $x = x_1$ Note vectors are in opposite directions at each end $x = x_0$ and $x = x_1$. 
Now we assume the motion is small i.e that \( |u_x| \) is small \( \Rightarrow \sqrt{1+u_x^2} \approx 1 \) (by Taylor's Exp.)

Then Eq. (1) says that \( T \) is constant along the string \( \Rightarrow T \) is independent of \( x \).

• Assume \( T \) is also independent of \( t \) (assumption)

Then Eq. (2) says:

\[
(T u_x)_x = \rho u_{tt}
\]

\[\Rightarrow \begin{array}{l}
  u_{tt} = c^2 u_{xx} \\
  c = \sqrt{\frac{T}{\rho}}
\end{array}
\]

\( (= \text{WAVE SPEED} \) )