Math 421 Cauchy-Riemann Eq'ns Polar Form 10/15/04

For a complex function f, consider its polar form

$$f\left(r\,e^{i\,\theta}\right) = U(r,\theta) + i\,V(r,\theta).$$

Theorem 1. ***If*** *f is differentiable at the point $z_0 = r_0\,e^{i\,\theta_0}$,*
then*:*

- *the first-order partial derivatives* $\dfrac{\partial U}{\partial r}, \dfrac{\partial U}{\partial \theta}, \dfrac{\partial V}{\partial r}, \dfrac{\partial V}{\partial \theta}$ *exist at* (r_0, θ_0)*, and*
- *at that* (r_0, θ_0) *they satisfy the Cauchy-Riemann conditions*

$$\begin{cases} \dfrac{\partial U}{\partial r} &= \;\;\dfrac{1}{r}\,\dfrac{\partial V}{\partial \theta} \\ \dfrac{\partial V}{\partial r} &= -\,\dfrac{1}{r}\,\dfrac{\partial U}{\partial \theta} \end{cases} \qquad (*)$$

Theorem 2. ***If***

- *f is continuous on a neighborhood of the point $z_0 = r_0\,e^{i\,\theta_0}$,*
- *the first-order partial derivatives* $\dfrac{\partial U}{\partial r}, \dfrac{\partial U}{\partial \theta}, \dfrac{\partial V}{\partial r}, \dfrac{\partial V}{\partial \theta}$ *exist on a neighborhood of* (r_0, θ_0)*,*
- *these partial derivatives are continuous at* (r_0, θ_0)*, and*
- *the Cauchy-Riemann conditions (*) are satisfied at* (r_0, θ_0)*,*

then *f is differentiable at z_0, and the derivative of f at that $z = z_0$ is given by*

$$\begin{aligned} f'(z) &= \quad e^{-i\,\theta}\left[\frac{\partial U}{\partial r} + i\,\frac{\partial V}{\partial r}\right] \\ &= \frac{1}{r}\,e^{-i\,\theta}\left[\frac{\partial V}{\partial \theta} - i\,\frac{\partial U}{\partial \theta}\right] \end{aligned}$$

where in these last formulas $(r,\theta) = (r_0,\theta_0)$ and the partial derivatives are all evaluated at $(r,\theta) = (r_0,\theta_0)$.