Math 421 • Fall 2006

Birth of complex numbers in solving cubic equations

Revised 9/6/06

The problem

To solve the cubic equation

$$x^3 + A x^2 + B x + K = 0.$$

Strategy:

- change-of-variable → new cubic with no quadratic term;
- solve new cubic;
- use the solutions of that to solve the original.

Reduction of cubic to depressed cubic (Anonymous, end of 14th century)

Temporarily replace *x* by *u* and rename the constant term K:

In[1]:= **cubic = u^3 + A u^2 + B u + K**

Out[1]= $K + B u + A u^2 + u^3$

Make **linear substitution**

$$u = x - \frac{1}{3} A$$

In[2]:= **depressed = cubic /. u → x − $\frac{1}{3}$ A**

Out[2]= $K + B\left(-\frac{A}{3} + x\right) + A\left(-\frac{A}{3} + x\right)^2 + \left(-\frac{A}{3} + x\right)^3$

Collect coefficients of the powers of x:

```
In[3]:= Collect[depressed, x]
```

Out[3]= $\frac{2\,A^3}{27} - \frac{A\,B}{3} + K + \left(-\frac{A^2}{3} + B\right) x + x^3$

That is the **depressed cubic:** no x^2 term, so of form

$$x^3 + b\,x + c$$

where

$$b = -\frac{A^3}{3} + B, \qquad c = \frac{2\,A^3}{27} - \frac{A\,B}{3} + K.$$

```
In[4]:= depressedCubic = x^3 + b x + c
```

Out[4]= $c + b\,x + x^3$

> **Exercise.** The linear substitution just used was $u = x - \frac{1}{3}A$. Among all possible linear substitutions $u = x -$ cst, why use cst $= \frac{1}{3}A$?

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del Ferro & Tartaglia solution of depressed cubic (Scipione del Ferro, 1515, and Niccolò Fontana aka "Tartaglia")

Scipione del Ferro and Niccolò Tartaglia discovered formula for a root of a depressed cubic $x^3 + b\,x + c$. In *Mathematica*:

In[5]:= `delFerroTartagliaRoot[b_, c_] :=` $\sqrt[3]{-\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}} + \sqrt[3]{-\frac{c}{2} - \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}}$

```
In[6]:= delFerroTartagliaRoot[b, c]
```

Out[6]= $\left(-\frac{c}{2} - \sqrt{\frac{b^3}{27} + \frac{c^2}{4}}\right)^{1/3} + \left(-\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{c^2}{4}}\right)^{1/3}$

The del Ferro-Tartaglia solution will appear simpler if in the depressed cubic you take $b = 3\,p$ and $c = 2\,q$ to obtain the form:

> $x^3 + 3\,p\,x + 2\,q$

```
In[7]:= delFerroTartagliaRoot[b, c] /. {b -> 3 p, c → 2 q}
```

Out[7]= $\left(-q - \sqrt{p^3 + q^2}\right)^{1/3} + \left(-q + \sqrt{p^3 + q^2}\right)^{1/3}$

Nicer formula from…

```
In[8]:= niceDepressedCubic = depressedCubic /. {b → 3 p, c → 2 q}
```

Out[8]= $2q + 3px + x^3$

…by new *Mathematica* function:

```
In[9]:= niceDelFerroTartagliaRoot[p_, q_] := delFerroTartagliaRoot[b, c] /. {b -> 3 p, c → 2 q}
```

So del Ferro-Tartaglia formula for root of a depressed cubic of form $x^3 + 3\,p\,x + 2\,q$ is:

```
In[10]:= niceDelFerroTartagliaRoot[p, q]
```

Out[10]= $\left(-q - \sqrt{p^3 + q^2}\right)^{1/3} + \left(-q + \sqrt{p^3 + q^2}\right)^{1/3}$

Cardan's general solution of the cubic
(Girolamo Cardano, *Ars magna*, 1545)

Cardan uses the linear substitution that reduces a general cubic to a depressed cubic along with the del Ferro-Tartaglia formula for one solution of the depressed cubic to obtain a general formula for solving any cubic. *Mathematica* can do it, too:

```
In[11]:= Solve[x^3 + A x^2 + B x + K == 0, x] // TraditionalForm
```

Out[11]//TraditionalForm=

$$\Bigg\{\Bigg\{x \to -\frac{A}{3} + \frac{1}{3\sqrt[3]{2}}\left(\left(-2A^3 + 9BA - 27K + 3\sqrt{3}\sqrt{4KA^3 - B^2A^2 - 18BKA + 4B^3 + 27K^2}\right)\text{^}(1/3)\right) - \left(\sqrt[3]{2}\,(3B - A^2)\right)\Big/\left(3\left(-2A^3 + 9BA - 27K + 3\sqrt{3}\sqrt{4KA^3 - B^2A^2 - 18BKA + 4B^3 + 27K^2}\right)\text{^}(1/3)\right)\Bigg\},$$

$$\Bigg\{x \to -\frac{A}{3} - \frac{1}{6\sqrt[3]{2}}\left(\left(1 - i\sqrt{3}\right)\left(-2A^3 + 9BA - 27K + 3\sqrt{3}\sqrt{4KA^3 - B^2A^2 - 18BKA + 4B^3 + 27K^2}\right)\text{^}(1/3)\right) + \left(\left(1 + i\sqrt{3}\right)(3B - A^2)\right)\Big/\left(3\,2^{2/3}\left(-2A^3 + 9BA - 27K + 3\sqrt{3}\sqrt{4KA^3 - B^2A^2 - 18BKA + 4B^3 + 27K^2}\right)\text{^}(1/3)\right)\Bigg\},$$

$$\Bigg\{x \to -\frac{A}{3} - \frac{1}{6\sqrt[3]{2}}\left(\left(1 + i\sqrt{3}\right)\left(-2A^3 + 9BA - 27K + 3\sqrt{3}\sqrt{4KA^3 - B^2A^2 - 18BKA + 4B^3 + 27K^2}\right)\text{^}(1/3)\right) + \left(\left(1 - i\sqrt{3}\right)(3B - A^2)\right)\Big/\left(3\,2^{2/3}\left(-2A^3 + 9BA - 27K + 3\sqrt{3}\sqrt{4KA^3 - B^2A^2 - 18BKA + 4B^3 + 27K^2}\right)\text{^}(1/3)\right)\Bigg\}\Bigg\}$$

Bombeli: use square-roots of negative numbers to obtain real roots of cubics
(Rafael Bombeli, *L'algebra*, 1572)

■ Paradox

• A depressed cubic *always* has a *real* solution: $x^3 + 3\,p\,x + 2\,q = 0$ is equivalent to

$$x^3 = -3\,p\,x - 2\,q,$$

and the cube function

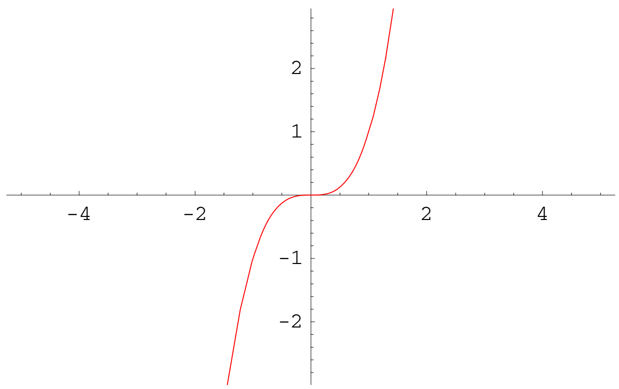
$$y = x^3$$

always intersects the line

$$y = -3\,p\,x - 2\,q$$

in at least one point, no matter what p and q are!

```
In[12]:= Plot[x^3, {x, -5, 5}, PlotStyle → Red];
```



• Yet del Ferro-Tartaglia formula $\left(-q - \sqrt{p^3 + q^2}\right)^{1/3} + \left(-q + \sqrt{p^3 + q^2}\right)^{1/3}$

involves square-roots of negative numbers when $q^2 < -p^3$.

■ Example

In the depressed cubic equation $x^3 + 3\,p\,x + 2\,q$, take $p = 5$ and $q = -2$:

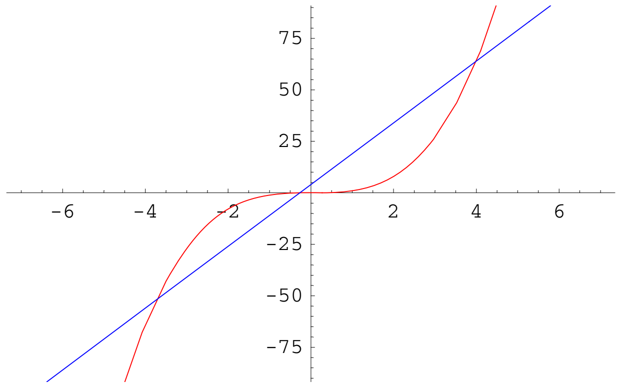
$$x^3 - 15\,x - 4 = 0$$

```
In[13]:= eqn = x^3 - 15 x - 4 == 0
```

```
Out[13]= -4 - 15 x + x^3 == 0
```

There *must* be a solution—the cubic $y = x^3$ and the line $y = 15x + 4$ must intersect:

```
In[14]:= Plot[{x^3, 15 x + 4}, {x, -7, 7}, PlotStyle → {Red, Blue}];
```



In fact, $x = 4$ is a solution:

```
In[15]:= eqn /. x → 4
```

```
Out[15]= True
```

■ Bombeli's "wild thought"

```
In[16]:= niceDelFerroTartagliaRoot[p, q]
```

$$\text{Out[16]=}\ \left(-q - \sqrt{p^3 + q^2}\right)^{1/3} + \left(-q + \sqrt{p^3 + q^2}\right)^{1/3}$$

The depressed cubic has $p = -5$, $q = -2$. So…

```
In[17]:= p^3 + q^2 /. {p → -5, q → -2}
```

```
Out[17]= -121
```

… then del Ferro-Tartaglia solution in this example is:

$$\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$$

The <u>known</u> solution $x = 4$ could be recovered from the del Ferro-Tartaglia *if* the two terms on the right have the forms

$$\sqrt[3]{2 + 11\sqrt{-1}} = m + n\sqrt{-1}, \quad \sqrt[3]{2 - 11\sqrt{-1}} = m - n\sqrt{-1}. \qquad (*)$$

for suitable m and n.

How? Well, the sum of these two solutions has the form…

In[18]:= $\left(m + n\sqrt{-1}\right) + \left(m - n\sqrt{-1}\right)$

Out[18]= $2\,m$

…so here we must have:

$2\,m = 4$ (the known solution).

Thus to obtain the solution $x = 4$, Bombeli wants to find n for which (*) holds with $m = 2$, in other words,

$$\left(2 + n\sqrt{-1}\right)^3 = 2 + 11\sqrt{-1}, \qquad \left(2 - n\sqrt{-1}\right)^3 = 2 - 11\sqrt{-1}$$

Pretend usual rules of algebra hold for expressions involving

$i = \sqrt{-1}$,

and **assume** the special rule

$i^2 = \left(\sqrt{-1}\right)^2 = -1.$

Then multiply out to calcluate $\left(2 + n\sqrt{-1}\right)^3$:

In[19]:= **theCube = Expand**$\left[\left(2 + n\sqrt{-1}\right)^3\right]$

Out[19]= $8 + 12\,i\,n - 6\,n^2 - i\,n^3$

To compare result with $2 + 11\sqrt{-1}$, separate the "real part" from the part involving i:

In[20]:= **ComplexExpand[theCube]**

Out[20]= $8 - 6\,n^2 + i\,(12\,n - n^3)$

That is supposed to equal $2 + 11\sqrt{-1}$:

In[21]:= **Solve**$[\{8 - 6\,n^2,\ 12\,n - n^3\} == \{2,\ 11\},\ n]$

Out[21]= $\{\{n \to 1\}\}$

Exercise. The "usual rules of algebra" include such identities as:

$(x + y) + (u + v) = (x + u) + (y + v),$

$(x + y)(u + v) = x\,u + y\,v + x\,v + y\,u,$

$k\,(x + y) = k\,x + k\,y,$

$k\,(x\,y) = (k\,x)\,y = x\,(k\,y),$

$(x + y) + z = x + (y + z).$

These identities hold for real numbers $x,\ y,\ u,\ v,\ k,\ z$.

Assume such identities hold for "complex numbers" as well—for numbers of the form $a + b\,i$ where a and b are real.

And still assume that $i^2 = i\,i = -1$. Then put each of the following into the form y=$u + i\,v$ with u and v real:

$(a + b\,i) + (c + d\,i),\ \ (a + b\,i)\,(c + d\,i)$

■ The moral

Square-roots of negative numbers are useful (essential?) in obtaining real roots of certain cubic equations.

(But what *are* such "complex" numbers? That's what's next in this course!)

Appendices

■ Appendix 1: Obtaining real imaginary parts of complex numbers involving symbolic quantities

How in *Mathematica* can you solve for *n* the equation

$$8 - 6n^2 + i(12n - n^3) = 2 + 11\sqrt{-1}$$

without copying and pasting, or reading off and retyping, the real and imaginary parts (as was done above)?

Use `Re` and `Im`.

In[22]:= **?Re**

    Re[z] gives the real part of the complex number z. More…

In[23]:= **?Im**

    Im[z] gives the imaginary part of the complex number z. More…

In[24]:= **theCube**

Out[24]= $8 + 12\,i\,n - 6\,n^2 - i\,n^3$

In[25]:= **Re[theCube]**

Out[25]= $8 - 12\,\mathrm{Im}[n] + \mathrm{Im}[n^3] - 6\,\mathrm{Re}[n^2]$

Try some more:

In[26]:= **Simplify[Re[theCube]]**

Out[26]= $8 - 12\,\mathrm{Im}[n] + \mathrm{Im}[n^3] - 6\,\mathrm{Re}[n^2]$

Stymied! Why? Because *Mathematica* doesn't know *n* should be real. Tell it so:

In[27]:= **ComplexExpand[Re[theCube]]**

Out[27]= $8 - 6\,n^2$

```
In[28]:= {realPart, imaginaryPart} = {ComplexExpand[Re[theCube]], ComplexExpand[Im[theCube]]}
```

Out[28]= $\{8-6\,n^2,\ 12\,n-n^3\}$

Now once again you could solve $8-6n^2+i(12n-n^3)=2+11\sqrt{-1}$:

```
In[29]:= Solve[{realPart, imaginaryPart} == {2, 11}, n]
```

Out[29]= $\{\{n\to 1\}\}$