Math 397C **10/1/01**

Definitions about attracting and repelling—corrected

Throughout, let $f\colon A \to A$ be a function from a subset A of $\mathbb{R}$ into itself.

For each nonnegative integer n, denote by f^n the nth **iterate** of f, so that also $f^n\colon A \to A$. Thus f^0 is the identity function of A; the first iterate $f^1 = f$; the second iterate $f^2 = f \circ f$; etc. Then for a point $x \in A$, the set $\{\, f^n(x) : n = 0, 1, 2, 3, \dots \}$ is the **orbit** of x under f.

Definition 1. An $x \in A$ is called a **fixed point** of f when $f(x) = x$.

If x is a fixed point of f, then $f^n(x) = x$ for every $n = 0, 1, 2, 3, \dots$ and so the orbit of x under f is just the one-point set $\{x\}$.

Definition 2. Let p be a fixed point of f.

Say that p **attracts** a point $x \in A$, and x is **attracted to** p when $\lim_{n\to\infty} f^n(x) = p$.

The **basin of attraction** of p is the set of all points $x \in A$ that are attracted to p.

The fixed point p (as well as its orbit $\{p\}$) is said to **attract**, and to be an **attractor**, when its basin of attraction includes $A \cap (p - \delta, p + \delta)$ for some $\delta > 0$. In other words, p is an attractor when all points of A that are sufficiently close to p are attracted to p.

Definition 3. (*Corrected!*) Let p be a fixed point of f. Then p (as well as its orbit $\{p\}$) is said to **repel**, and to be a **repellor**, when, for some $\delta > 0$, for each $x \in A \cap (p - \delta, p + \delta)$ with $x \neq p$, there is at least one power n such that $f^n(x) \notin (p - \delta, p + \delta)$. In other words, p repels when, for some $\delta > 0$, the orbit of each point $x \in A \cap (p - \delta, p + \delta)$ (other than of p itself) does *not* remain in $(p - \delta, p + \delta)$.

Definition 4. A point $p \in A$ is said to be a **periodic point**—and its orbit is said to be a **periodic orbit**—if there is some integer $k \geq 2$ for which $f^k(p) = p$. In this case the least such k is called the **(prime) period** of p.

According to the preceding definition, a fixed point is *not* considered to be periodic. Some authors do so consider it. In any case, you could regard a fixed point as a sort of "degenerate" case of a periodic point.

Suppose p is a periodic point of f with period k. The also $f^{k+1}(p) = f(p), f^{k+2}(p) = f^2(p)$, etc. Thus the entire orbit of p reduces to just the finite set $\{p, f(p), f^2(p), \dots, f^{k-1}(p)\}$ consisting of exactly k distinct points.

If p is a periodic point of f with period k, then p is a fixed point of the kth iterate $f^k\colon A \to A$. In this case we may consider the new

Definition 5. Let p be a *periodic* point of f with period k. Consider instead of f the function $f^k\colon A \to A$. Say that p **attracts** or **repels** when p attracts or repels, respectively, for f^k. In this situation, also call the periodic orbit of p under f a **periodic attractor** or **periodic repellor**, respectively.