Math 331.1 **Exam 3 Answers** **December 9, 2005**

1. (a) [15%] The characteristic equation

$$s^2 + 5s + 4 = 0, \qquad \text{that is,} \quad (s+1)(s+4) = 0,$$

has distinct real roots

$$s = -1, \qquad s = -4.$$

Thus no "second-guessing" is needed—that is, no multiplication by t—to find a solution. [1%] [Note that, since (a) asks just for *some* solution, you do *not* need to add in here the general solution of the associated homogeneous solution!]

First method: complexify, then descend to the real world. First, find a solution y_c of the complexified ODE

$$y'' + 5\,y' + 4\,y = 34\,e^{i\,t}. \qquad [1\%] FORM$$

Guess

$$y_c = Ae^{i\,t}. \qquad [2\%] FORM$$

Then

$$y_c{}' = Aie^{i\,t}, \qquad y_c{}'' = -Ae^{i\,t}. \qquad [2\%] PLUG$$

That y_c is a solution of the complexified ODE means

$$-Ae^{i\,t} + 5Aie^{i\,t} + 4Ae^{i\,t} = 34e^{i\,t}, \qquad [2\%] PLUG$$

for all t, or equivalently,

$$(3 + 5i)A = 34. \qquad [2\%] startCALC$$

The desired A is thus

$$A = \frac{34}{3+5i} = \frac{34}{3+5i} \cdot \frac{3-5i}{3-5i} = 3 - 5i,$$

and so the complexified ODE has as a solution

$$y_c(t) = (3 - 5i)e^{i\,t}. \qquad [1\%] CALC$$

Thus

$$y_c(t) = (3 - 5i)(\cos t + i \sin t) = (3\cos t + 5\sin t) + i(3\sin t - 5\cos t), \qquad [1\%] CALC$$

Since the factor $\sin t$ of the forcing function is the *imaginary* part of $e^{i\,t}$, the imaginary part y_{im} of y_c will be a real solution y_p of the given ODE: [2%] FORM

$$\boxed{y_p(t) = y_{\text{im}}(t) = 3\sin t - 5\cos t.} \qquad [1\%] CALC$$

Second method: match the form in the real world. This method is based upon the result in general obtained from complexifying, without actually ascending to the complex world. First, "guess" a solution of the form

$$y_p(t) = A\,\cos t + B\,\sin t. \qquad [4\%] FORM$$

(You do *not* know that A must be 0.) Then

$$y_p{}' = -\,A\,\sin t + B\,\cos t, \qquad y_p{}'' = -\,A\,\cos t - B\,\sin t. \qquad [1\%] PLUG$$

That y_p is a solution of the given ODE means

$$(-\,A\,\cos t - B\,\sin t) + 5\,(-\,A\,\sin t + B\,\cos t) + 4\,(-\,A\,\sin t + B\,\cos t) = 34\,\sin t,$$

that is,

$$(3\,A+5\,B)\,\cos t+(-\,5\,A+3\,B)\,\sin t=34\,\sin t. \qquad [4\%] PLUG$$

Use linear independence of sin and cos (or substitute first $t=0$ and then $t=\pi/2$) to obtain:

$$\begin{cases} 3\,A+5\,B=0, \\ -\,5\,A+3\,B=34. \end{cases} \qquad [3\%] startCALC$$

Then

$$A=-\,5, \qquad B=3. CALC$$

Thus the desired solution, by this method too, is:

$$\boxed{y_p(t)=3\sin t-5\cos t.} \qquad [2\%] CALC$$

(b) [5%] From work in (a), the general solution of the associated unforced (homogeneous) ODE is

$$y_h(t)=k_1e^{-t}+k_2e^{-4t}. \qquad [2\%]$$

Then the general solution of the given ODE—that is, the form of all solutions—is :

$$y(t)=y_h(t)+y_p(t) \qquad [2\%]$$

Thus:

$$\boxed{y(t)=k_1e^{-t}+k_2e^{-4t}+3\sin t-5\cos t.} \qquad [1\%]$$

2. *Step 1: Take Laplace transforms.* [6%] Let $Y=\mathcal{L}[y]$. Then:

$$\mathcal{L}[y'] + 2\,\mathcal{L}[y] = 3\,\mathcal{L}[e^{-\,6\,t}],$$
$$sY-y(0)+2\,Y=3\frac{1}{s-(-6)},$$
$$sY-5+2\,Y=3\frac{1}{s+6}.$$

Step 2: Solve for the Laplace transform Y of y: [2%]

$$(s+2)Y-5=\frac{3}{s+6},$$
$$Y=\frac{5}{s+2}+\frac{3}{(s+2)\,(s+6)}$$

Step 3: Use partial fractions on the right-hand-side: [5%] (You may use your calculator's `expand` function if you wish.)

$$\frac{3}{(s+2)\,(s+6)}=\frac{A}{s+2}+\frac{B}{s+6},$$
$$3=A\,(s+6)+B\,(s+2),$$
$$s=-2: \quad 3=A(4)+0 \implies A=3/4,$$
$$s=-6: \quad 3=0+B(-4) \implies B=-\,3/4,$$
$$\frac{3}{(s+2)\,(s+6)}=\frac{3/4}{s+2}+\frac{-\,3/4}{s+6}$$

Then

$$
\begin{aligned}
Y &= \frac{5}{s+2} + \frac{3/4}{s+2} + \frac{-3/4}{s+6} \\
&= \frac{23/4}{s+2} - \frac{3/4}{s+6}
\end{aligned}
$$

Step 4: Take inverse Laplace transforms: [7%]

$$
\begin{aligned}
y(t) = \mathcal{L}^{-1}[Y] &= \frac{23}{4}\,\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \frac{3}{4}\,\mathcal{L}^{-1}\left[\frac{1}{s+6}\right] \\
&= \frac{23}{4}\,\mathcal{L}^{-1}\left[\frac{1}{s-(-2)}\right] - \frac{3}{4}\,\mathcal{L}^{-1}\left[\frac{1}{s-(-6)}\right] \\
&= \frac{23}{4}\,e^{-2t} - \frac{3}{4}\,e^{-6t}
\end{aligned}
$$

Thus the solution is:

$$
\boxed{y(t) = \frac{23}{4}\,e^{-2t} - \frac{3}{4}\,e^{-6t}.}
$$

3. (a) [5%] *First method: match the form.* The form of the system is $\vec{Y}\,' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \vec{Y}$, which is the system equivalent to the single 2nd-order ODE $y\,'' + p\,y\,' + q\,y = 0$. In the present case, this ODE is therefore

$$
\boxed{y\,'' + 2\,y\,' + y = 0.}
$$

Second method: eliminate a variable. Write $\vec{Y} = \begin{bmatrix} y \\ v \end{bmatrix}$. The given vector ODE represents the system

$$
\begin{cases} y\,' = v, \\ v\,' = -y - 2\,v. \end{cases}
$$

Then

$$
y\,'' = v\,' = -y - 2\,v = -y - 2\,y\,',
$$

which yields same ODE $y\,'' + 2\,y\,' + y = 0$ as did the first method.

(b) [15%] *First method: Solve the 2nd-order ODE.* The characteristic equation of the equivalent 2nd-order ODE is $s^2 + 2s + 1 = 0$, that is, $(s+1)^2 = 0$, which has the **repeated** root

$$
s = -1. \qquad [2\%]
$$

The general solution of the 2nd-order ODE for y is therefore

$$
y = k_1\,e^{-t} + k_2\,t\,e^{-t}. \qquad [4\%]
$$

Then

$$
v = y\,' = -k_1\,e^{-t} + k_2\,(-t\,e^{-t} + 1\,e^{-t}) = -k_1\,e^{-t} + k_2\,(1-t)\,e^{-t}. \qquad [3\%]
$$

The corresponding solution $\vec{Y} = \begin{bmatrix} y \\ v \end{bmatrix}$ of the system is

$$
\boxed{\vec{Y} = \begin{bmatrix} k_1\,e^{-t} + k_2\,t\,e^{-t} \\ -k_1\,e^{-t} + k_2\,(1-t)\,e^{-t} \end{bmatrix} = e^{-t} \begin{bmatrix} k_1 + k_2 t \\ -k_1 + k_2\,(1-t) \end{bmatrix},} \qquad [6\%]
$$

and so this is the general solution of the system.

Second method: use eigenvalues and eigenvectors to solve the system. Call the coefficient matrix A. Then

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix},$$

Find eigenvalues of A. The characteristic polynomial is

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = -\lambda(-2-\lambda) - (-1) = \lambda^2 + 2\lambda + 1,$$

which has the repeated root $\lambda = -1$. Thus A has the **repeated** eigenvalue

$$\lambda_1 = -1. \qquad \textbf{[2\%]}$$

Find an eigenvector $\vec{V_1} = \begin{bmatrix} y_1 \\ v_1 \end{bmatrix}$ corresponding to this eigenvalue. Such an eigenvector satisfies

$$(A - \lambda_1 I)\vec{V_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \text{that is,} \qquad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Equivalently, the components of $\vec{V_1}$ satisfy $y_1 + v_1 = 0$, that is, $v_1 = -y_1$. Take, say, $v_1 = 1$—other choices *except* 0 are OK, too—to get the eigenvector

$$\vec{V_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \qquad \textbf{[4\%]}$$

Find a "generalized eigenvector" $\vec{V_2} = \begin{bmatrix} y_2 \\ v_2 \end{bmatrix}$ from $\vec{V_2}$. Such a $\vec{V_2}$ satisfies

$$(A - \lambda_1 I)\vec{V_2} = \vec{V_1}, \qquad \text{that is,} \qquad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Equivalently, the components of $\vec{V_2}$ satisfy $y_2 + v_2 = 1$. Take, say, $y_2 = 0$—other choices are OK, too—to get the generalized eigenvector

$$\vec{V_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \qquad \textbf{[4\%]}$$

The general solution of the system is

$$\vec{Y}(t) = k_1\vec{Y_1}(t) + k_2\vec{Y_2}(t) \qquad \textbf{[1\%]}$$

where

$$\vec{Y_1}(t) = e^{\lambda_1 t}\vec{V_1} = e^{-t} = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$\vec{Y_2}(t) = e^{\lambda_1 t}(t\vec{V_1} + \vec{V_2}) = e^{-t}\left(t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = e^{-t}\begin{bmatrix} t \\ -t+1 \end{bmatrix}. \qquad \textbf{[3\%]}$$

Thus

$$\vec{Y}(t) = k_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + k_2 e^{-t}\begin{bmatrix} t \\ -t+1 \end{bmatrix} = e^{-t}\begin{bmatrix} k_1 + k_2 t \\ -k_1 + k_2(1-t) \end{bmatrix}. \qquad \textbf{[1\%]}$$

[This is, of course, the same general solution as from the first method. (The form you get may differ according to which eigenvector and generalized eigenvector you chose.)]

Third method: textbook's version of second method. Again call the coefficient matrix A. Then

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix},$$

As in the second method, there is only a single eigenvalue, $\lambda_1 = -1$. [2%] The general solution has the form (with *no* k_1 or k_2!)

$$\vec{Y}(t) = e^{-t}\,\vec{V_0} + t\,e^{-t}\,\vec{V_1} \qquad \textbf{[6\%]}$$

where $\vec{V_0} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is an arbitrary vector ([2%]) and

$$\vec{V_1} = (A - \lambda\, I)\,\vec{V_0} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 + y_0 \\ -\,x_0 -\, y_0 \end{bmatrix} \qquad \textbf{[5\%]}$$

Then the general solution is

$$\boxed{\vec{Y}(t) = e^{-t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t\,e^{-t} \begin{bmatrix} x_0 + y_0 \\ -\,x_0 -\, y_0 \end{bmatrix}}$$

4. (a) [6%]

- *Case $k = 0$:* In this case, where there is no damping whatsoever, the equation describes simple harmonic motion—a sinusoidal function with some constant amplitude. So in this case the solution curves $y(t)$ do *not* tend toward 0 as $t \to \infty$. Hence the trajectories in the phase plane do *not* tend toward the origin. [2%]
- *Case $k > 0$:* In this case, where there is damping, every solution curve $y(t) \to 0$ as $t \to \infty$, and $y'(t) \to 0$ as $t \to \infty$. Hence all trajectories tend toward the origin when $k > 0$. [4%]

(b) [14%] Assume $k > 0$. (2%) The characteristic equation $s^2 + ks + 4$ has roots

$$s = \frac{-k \pm \sqrt{k^2 - 16}}{2}. \qquad \textbf{[1\%]}$$

- *Underdamping: $k^2 - 16 < 0$, that is, $0 < k < 4$.* In this case the characteristic equation has conjugate complex roots $-\frac{k}{2} \pm i\frac{\sqrt{16-k^2}}{2}$. The origin is a "spiral sink" in the phase plane. This means that, as they tend toward the origin, *the trajectories forever spiral around and the origin and toward it.* [4%]
- *Overdamping: $k^2 - 16 > 0$, that is, $k > 4$.* In this case the characteristic equation has distinct real (negative) roots. The origin is a non-spiral sink. This means that, as they tend toward the origin, *the trajectories do not spiral around the origin but, instead, just head toward it (along a curved path).* [4%]
- *Critical damping: $k^2 - 16 = 0$, that is, $k = 4$.* (This was the example of critical damping demonstrated in class.) In this case the characteristic equation has the repeated negative real root $-\frac{k}{2} = -2$. As they tend toward the origin, *the trajectories do not spiral around the origin but, instead, head toward it along a curved path.* [4%] This path is *tangent to the unique straight-line trajectory that lies on the line of eigenvectors (corresponding eigenvectors.* [1%]

(These behaviors are, of course, for all trajectories *except* the equilibrium point at the origin.)

In each case, you could give your answer by sketching trajectories in the phase plane instead of using words such as the above.

5. That $y_1(t)$ and $y_2(t)$ are solutions of the nonhomogeneous ODE means

$$y_1{}'' + p\,y_1{}' + q\,y_1 = e^{-8t}\cos(7\pi t),$$
$$y_2{}'' + p\,y_2{}' + q\,y_2 = e^{-8t}\cos(7\pi t). \qquad \textbf{[5\%]}$$

Then

$$\begin{aligned} y_0{}'' + p\,y_0{}' + q\,y_0 &= (y_1 - y_2)'' + p\,(y_1 - y_2)' + q\,(y_1 - y_2) && \textbf{[2\%]} \\ &= (y_1{}'' - y_2{}'') + p\,(y_1{}' - y_2{}') + q\,(y_1 - y_2) && \textbf{[2\%]} \\ &= (y_1{}'' + p\,y_1{}' + q\,y_1) - (y_2{}'' + p\,y_2{}' + q\,y_2) && \textbf{[4\%]} \\ &= e^{-8t}\cos(7\pi t) - e^{-8t}\cos(7\pi t) && \textbf{[4\%]} \\ &= 0. && \textbf{[3\%]} \end{aligned}$$

Thus y_0 is a solution of the associated homogeneous ODE $y'' + p\,y' + q\,y = 0$.

6. The ODE has the form $x'' + {\omega_0}^2\,x = \sin(\alpha t)$ with $\omega_0 = 2$. The forcing frequency is $\alpha/(2\pi)$. The characteristic equation $s^2 + 4 = 0$ has roots $s = \pm 2i$, and so the associated unforced system has natural frequency $\omega_0/(2\pi) = 1/\pi$. **[4%]**

- *Case $\alpha = \omega_0 = 2$:* **[7%]** This is the case of **resonance**, when the frequency of the forcing function equals the natural frequency. As $t \to \infty$, the solution $x(t)$ oscillates with every-increasing amplitude, in fact, with an amplitude proportional to t. (You could sketch this behavior.)
- *Case $\alpha \neq \omega_0 = 2$:* **[2%]** In general, when the frequency $\alpha/(2\,\pi)$ of the forcing function is different from the natural frequency $\omega_0/(2\pi) = 1/\pi$, the solution is the superposition of the unforced response

  $$x_h(t) = k_1\cos 2t + k_2\sin 2t$$

  of period $2\,\pi/\omega_0 = \pi$ and the forced response

  $$x_p(t) = \frac{1}{4 - \alpha^2}\sin\alpha t$$

  of period $2\pi/\alpha$. For some values of t, the forced and unforced responses have the same sign and reinforce one another; for other values of t, the forced and unforced responses have opposite signs and cancel each other out. (There's not a whole lot to say about this general case. You could draw one or more sketches of such solutions.)
- *Case $\alpha \neq \omega_0 = 2$ but $\alpha \approx \omega_0 = 2$:* **[7%]** This is the case of **beats**, when the frequency of the forcing function is close to the natural frequency. The responses of $x(t)$ to the two frequencies tend to reinforce or cancel each other out over long time intervals. This means that—at least in the case $x(0) = x'(0) = 0$—the graph of $x(t)$ is a rapidly oscillating sinusoidal function enveloped by a slowly oscillating sinusoidal function.[1] (You could sketch this behavior.)

[1] You were *not* expected to justify that assertion by noting that. in the case $x(0) = x'(0) = 0$:

$$x(t) = \frac{2}{\alpha^2 - 4}\left(\sin\frac{\alpha - 2}{2}\,t\right)\cdot\left(\sin\frac{\alpha + 2}{2}\,t\right)$$