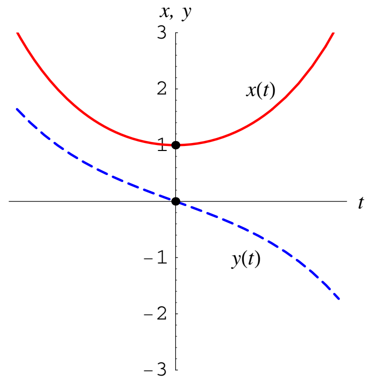
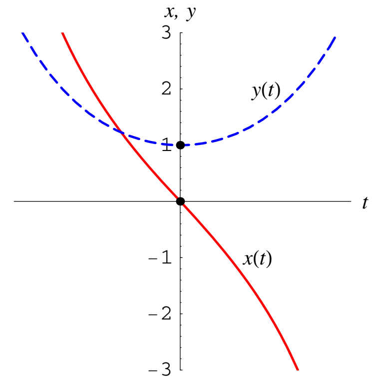
Math 331.1 **Exam 2 Answers** **November 9, 2005**

1. [$4 \times 5\%$] {*How I drew the solution curves:* From the ODEs, $x'(t)$ and $y'(t)$ are *negative* multiples of $y(t)$ and $x(t)$, respectively; then the general direction of travel along the first solution curve [for $x(0) = 1$, $y(0) = 0$] is **right-to-left**, and the general direction of travel along the second solution curve [for $x(0) = 0$, $y(0) = 1$] is **bottom-to-top**. The second trajectory crosses the line $y = x$ at some time $t_1 < 0$, so the graphs of $x(t)$ and $y(t)$ intersect when $t = t_1$.}



$x(0) = 1$ and $y(0) = 0$



$x(0) = 0$ and $y(0) = 1$

2. (a) [15%] From

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{bmatrix}$$

obtain the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = (2-\lambda)(4-\lambda) - 3 = (\lambda^2 - 6\lambda + 8) - 3 = \lambda^2 - 6\lambda + 5. \qquad [2\%]$$

Then

$$p(\lambda) = (\lambda - 1)(\lambda - 5)$$

so that the eigenvalues of A are

$$\lambda_1 = 1, \qquad \lambda_2 = 5. \qquad [2\%]$$

Find an eigenvector for $\lambda_1 = 1$ by solving $(A - \lambda_1 I)\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{cases} x_1 + y_1 = 0 \\ 3x_1 + 3y_1 = 0 \end{cases} \implies y_1 = -x_1$$

Take, say, $x_1 = 1$, so that $y_1 = -1$. Thus an eigenvector for $\lambda_1 = 1$ is $\mathbf{V}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. [2%]

Find an eigenvector for $\lambda_2 = 5$ by solving $(A - \lambda_1 I)\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{cases} -3x_2 + y_2 = 0 \\ 3x_2 - y_2 = 0 \end{cases} \implies y_2 = 3x_2$$

Take, say, $x_2 = 1$, so that $y_2 = 3$. Thus an eigenvector for $\lambda_2 = 5$ is $\mathbf{V}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. [2%]

Two solutions of the given system of ODEs are:

$$\mathbf{Y}_1(t) = e^{\lambda_1 t}\mathbf{V}_1 = e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}, \qquad \mathbf{Y}_2(t) = e^{\lambda_2 t}\mathbf{V}_2 = e^{5t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} e^{5t} \\ 3e^{5t} \end{bmatrix} \qquad [2\%]$$

{Since A is 2×2 and has 2 distinct eigenvalues, the theory guarantees that the vectors $\mathbf{Y}_1(0) = \mathbf{V}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{Y}_2(0) = \mathbf{V}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are linearly independent.}

Hence the general solution is:

$$\boxed{\mathbf{Y}(t) = k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t) = k_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + k_2 e^{5t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} k_1 e^t + k_2 e^{5t} \\ -k_1 e^t + 3k_2 e^{5t} \end{bmatrix}} \qquad [3\%]$$

(b) [5%] The initial condition $\mathbf{Y}(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ gives

$$\begin{bmatrix} k_1 + k_2 \\ -k_1 + 3k_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \qquad \text{which means} \qquad \begin{cases} k_1 + k_2 = 5 \\ -k_1 + 3k_2 = 3. \end{cases} \qquad [2\%]$$

Add the two equations to obtain $4k_2 = 8$. Then

$$k_2 = 2, \qquad k_1 = 5 - k_2 = 5 - 2 = 3. \qquad [1\%]$$

Thus the desired solution is:

$$\boxed{\mathbf{Y}(t) = 3\mathbf{Y}_1(t) + 2\mathbf{Y}(t) = 3e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2e^{5t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3e^t + 2e^{5t} \\ -3e^t + 6e^{5t} \end{bmatrix}} \qquad [2\%]$$

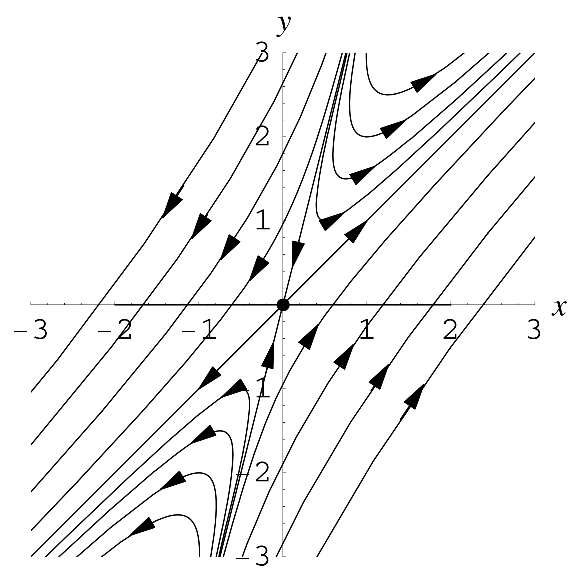
3. (a) **[8%]** The solution involved is

$$Y(t) = e^{\lambda_1 t}\mathbf{V}_1 = e^{-t}\begin{bmatrix}1\\4\end{bmatrix}. \qquad \textbf{[2\%]}$$

Its trajectory in the (x, y)-phase plane is the *part* of the line through $(0, 0)$ and $(1, 4)$ lying in the first quadrant—exclusive of the origin. That is, it is the open ray from the origin and passing through $(1, 4)$. **[4%]**

The direction along this curve is **inward**, toward the origin. **[2%]**

(b) **[12%]** {Since the eigenvalues are real with $\lambda_1 < 0 < \lambda_2$, the origin is a *saddle point*. The system has straight line solutions along lines through the origin and each of the given eigenvectors. Along the line having the direction of $\mathbf{V}_1$, the direction of flow is *toward* the origin (because the corresponding eigenvalue $\lambda_1 < 0$). Along the line having the direction of $\mathbf{V}_2$, the direction of flow is *away from* the origin (because the corresponding eigenvalue $\lambda_2 > 0$). This information is sufficient to determine the qualitative picture for the entire phase portrait; just keep the directions on the other solution curves consistent with those along the two special lines.}



4. (a) **[12%]** Variable

   - y represents the predator population, and
   - x represents the prey population. **[6%]**

   The reason is that interactions between the two populations, as represented by the product $x\,y$, increase y' but decrease x' (since the sign before $x\,y$ is positive in the expression for y' whereas the sign before $x\,y$ is negative in the expression for x'). **[6%]**

   (b) **[8%]** The two species are *competitive*: they compete for the same food, territory, etc. **[4%]**

   The reason is that interactions between the two populations, as represented by the product $x\,y$, decrease both x' and y' (since the sign before $x\,y$ is negative in the expressions for both x' and y'). **[4%]**

5. (a) [8%] Let $v = x'$. [2%] Then

$$v' = x'' = -5x - 2x' = -5x - 2v. \qquad [3\%]$$

Thus the equivalent system of first-order ODEs is:

$$\boxed{\begin{cases} x' = & v \\ v' = & -5x - 2v \end{cases}} \qquad [3\%]$$

This could be written in matrix form as

$$\mathbf{Y}' = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \mathbf{Y} \qquad \text{where} \qquad \mathbf{Y} = \begin{bmatrix} x \\ v \end{bmatrix}.$$

(b) [12%] Either of two methods may be used.

Method 1: guess-and-check for given 2nd-order ODE.

Try a solution of the form

$$x(t) = e^{st} \qquad [2\%]$$

and determine the constant s. From theory, s is a root of the quadratic equation

$$s^2 + 2s + 5 = 0. \qquad [2\%]$$

The roots of this equation are

$$s = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i. \qquad [2\%]$$

Use $s = -1 + 2i$—which is complex—to get from Euler's formula the complex solution

$$x(t) = e^{(-1+2i)t} = e^{-t}(\cos 2t + i \sin 2t). \qquad [2\%]$$

The real and complex parts of this solution are also solutions; these are

$$x_1(t) = e^{-t} \cos 2t, \qquad x_2(t) = e^{-t} \sin 2t. \qquad [2\%]$$

Easy way to obtain the desired solution from $x_1(t)$ and $x_2(t)$: Notice that $x_2(0) = 0$; further,

$$x_2'(t) = 2e^{-t} \cos 2t - e^{-t} \sin 2t,$$

so that $x_2'(0) = 2$. Hence the desired solution is $x(t) = x_2(t)$, that is:

$$\boxed{x(t) = e^{-t} \sin 2t} \qquad [2\%]$$

Alternate way to obtain the desired solution form $x_1(t)$ and $x_2(t)$: The general solution is

$$x(t) = k_1\, x_1(t) + k_2\, x_2(t) = k_1\, e{-}t\, \cos 2t + k_2\, e^{-t}\, \sin 2t.$$

The initial condition $x(0) = 0$ gives $k_1 = 0$, so what's left is

$$x(t) = k_2\, e^{-t}\, \sin 2t.$$

Then $x'(t) = 2\, k_2\, e^{-t}\, \sin 2t$. The initial condition $x'(0) = 2$ now gives $2\, k_2 = 2$, that is, $k_2 = 1$. Hence the desired solution is $x(t) = x_2(t) = e^{-t}\, \sin 2t$.

Method 2: eigenvalue/eigenvector method for system of 1st-order ODEs.
The equivalent system of 1st-order ODEs is $\mathbf{Y}' = A\mathbf{Y}$ where $A = \left[\begin{smallmatrix} 0 & 1 \\ -5 & -2 \end{smallmatrix}\right]$. From $A - \lambda I = \left[\begin{smallmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{smallmatrix}\right]$ obtain the characteristic polynomial

$$\det(A - \lambda I) = -\lambda(-2 - \lambda) - (-5) = \lambda^2 + 2\lambda + 5$$

whose roots are

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

so that the eigenvalues of A are

$$\lambda_1 = -1 + 2i, \qquad \lambda_2 = -1 - 2i. \qquad \textbf{[3\%]}$$

Find an eigenvector for $\lambda_1 = -1 + 2i$ by solving $(A - \lambda_1 I)\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$\begin{bmatrix} 1 - 2i & 1 \\ -5 & -1 - 2i \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{cases} (1 - 2i)x_1 + \qquad\qquad y_1 = 0 \\ \qquad -5x_1 + (-1 - 2i)y_1 = 0 \end{cases}$$

(These two linear algebraic equations here are redundant; the second is $-1 - 2i$ times the first.) The general solution of these two linear algebraic equations is

$$y_1 = (-1 + 2i)x_1.$$

Take, say, $x_1 = 1$, so that $y_1 = -1 + 2i$. Thus an eigenvector corresponding to $\lambda_1 = -1 + 2i$ is

$$\mathbf{V}_1 = \begin{bmatrix} 1 \\ -1 + 2i \end{bmatrix}. \qquad \textbf{[2\%]}$$

Thus a complex solution of the system of ODEs is

$$\begin{aligned} Y_c(t) &= e^{-\lambda_1 t}\mathbf{V}_1 = e^{(-1+2i)t}\begin{bmatrix} 1 \\ -1 + 2i \end{bmatrix} = e^{-t}e^{2ti}\begin{bmatrix} 1 \\ -1 + 2i \end{bmatrix} \\ &= e^{-t}(\cos 2t + i \sin 2t)\begin{bmatrix} 1 \\ -1 + 2i \end{bmatrix} = e^{-t}\begin{bmatrix} \cos 2t + i \sin 2t \\ (-\cos 2t - 2\sin 2t) + i(2\cos 2t - \sin 2t) \end{bmatrix} \\ &= e^{-t}\begin{bmatrix} \cos 2t \\ -\cos 2t - 2\sin 2t \end{bmatrix} + ie^{-t}\begin{bmatrix} \sin 2t \\ 2\cos 2t - \sin 2t \end{bmatrix} \qquad \textbf{[3\%]} \end{aligned}$$

Then the system also has as real-valued solutions the real and imaginary parts

$$\mathbf{Y}_{\text{re}}(t) = e^{-t}\begin{bmatrix} \cos 2t \\ -\cos 2t - 2\sin 2t \end{bmatrix}, \qquad \mathbf{Y}_{\text{im}}(t) = e^{-t}\begin{bmatrix} \sin 2t \\ 2\cos 2t - \sin 2t \end{bmatrix}. \qquad \textbf{[2\%]}$$

Easy way to obtain the desired solution from $\mathbf{Y}_{\text{re}}(t)$ and $\mathbf{Y}_{\text{im}}(t)$: Since $\mathbf{Y}_{\text{im}}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, immediately

$$\begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \mathbf{Y}_{\text{im}}(t) = e^{-t}\begin{bmatrix} \sin 2t \\ 2\cos 2t - \sin 2t \end{bmatrix}$$

yields the desired solution

$$\boxed{x(t) = e^{-t}\sin 2t} \qquad \textbf{[2\%]}$$

Alternate way to obtain the desired solution from $\mathbf{Y}_{\text{re}}(t)$ *and* $\mathbf{Y}_{\text{im}}(t)$*:* The general solution of the system of ODEs is

$$\begin{aligned}\mathbf{Y}(t) &= k_1\mathbf{Y}_{\text{re}}(t) + k_2\mathbf{Y}_{\text{im}}(t) = k_1 e^{-t}\begin{bmatrix}\cos 2t\\ -\cos 2t - 2\sin 2t\end{bmatrix} + k_2 e^{-t}\begin{bmatrix}\sin 2t\\ 2\cos 2t - \sin 2t\end{bmatrix}\\ &= e^{-t}\begin{bmatrix}k_1\cos 2t + k_2\sin 2t\\ k_1(-\cos 2t - 2\sin 2t) + k_2(2\cos 2t - \sin 2t)\end{bmatrix}.\end{aligned}$$

The initial condition is

$$\mathbf{Y}(0) = \begin{bmatrix}x(0)\\ v(0)\end{bmatrix} = \begin{bmatrix}x(0)\\ x'(0)\end{bmatrix} = \begin{bmatrix}0\\ 2\end{bmatrix}.$$

Then

$$\begin{cases} k_1 &= 0\\ -k_1 + 2k_2 &= 2,\end{cases} \qquad \Longrightarrow \qquad k_1 = 0, \qquad k_2 = 1.$$

Thus the desired solution to the 1st-order system is

$$\mathbf{Y}(t) = 0\mathbf{Y}_{re}(t) + 1\mathbf{Y}_{im}(t) = e^{-t}\begin{bmatrix}\sin 2t\\ 2\cos 2t - \sin 2t\end{bmatrix} = \begin{bmatrix}e^{-t}\sin 2t\\ e^{-t}(2\cos 2t - \sin 2t)\end{bmatrix}.$$

This means that the desired solution to the given 2nd-order ODE is:

$$\boxed{x(t) = e^{-t}\sin 2t}$$

6. (a) [8%] Equilibrium points of the given system of nonlinear ODEs are the solutions (x, y) of the system of algebraic equations:

$$\begin{cases} x(5-x-y)=0 \\ y(-2+x)=0 \end{cases} \qquad \textbf{[2\%]} \qquad (*)$$

To solve these equations, consider several cases:
Case 1: $x \neq 0$ and $y \neq 0$. In this case, equilibrium points are solutions of

$$\begin{cases} 5-x-y=0 \\ -2+x=0, \end{cases}$$

in other words, $x = 2$ and so $y = 5 - x = 5 - 2 = 3$. Thus $(2, 3)$ is an equilibrium point.
Case 2: $x \neq 0$ but $y = 0$. In this case, equations (*) reduce to the single equation

$$x(5-x)=0,$$

which has solutions $x = 0$, $x = 5$. The solution $x = 0$ is inconsistent with the assumption in this case that $x \neq 0$. Hence the only relevant solution is $x = 5$. Thus $(5, 0)$ is an equilibrium point.
Case 3: $x = 0$ but $y \neq 0$. In this case, equations (*) reduce to the single equation

$$-2=0$$

which has no solutions. Thus this case provides no equilibrium points.
Case 4: $x = 0$ and $y = 0$. This case provides the equilibrium point $(0, 0$.
Thus there are three equilibrium points:

$$\boxed{(0,0), \quad (5,0), \quad (2,3)} \qquad \textbf{[6\%]}$$

(b) **[12%]** {*How I selected the points and drew the trajectory:* Notice that x starts for $t = 0$ at $x = 3$, rises to a maximum ≈ 4 at a time when $y \approx 1.2$, falls to a minimum ≈ 1.8 when $y \approx 3.2$, rises to a local maximum ≈ 2.1 when $y \approx 2.9$, and finally levels out roughly to 2. Likewise, y starts for $t = 0$ at $y = 0.5$, rises to a maximum ≈ 3.5 when $x \approx 2$, falls to a local minimum ≈ 2.8 when $x \approx 2$, rises to a local maximum ≈ 3.1 when $x \approx 2.1$, and finally levels out roughly to 3. These are the points marked on the trajectory below, with the corresponding points marked on the graphs of $x(t)$ and $y(t)$. Thus, as shown below, as $t \to \infty$ the point $(x(t), y(t)$ **spirals toward and counterclockwise around the equilibrium point** $(2, 3)$ —but does not actually reach it.}