Math 331.1 **Exam 1 Answers** **Oct. 7, 2005**

1. (20%) The ODE is separable. Because $y(0) = 3$, ignore the equilibrium solution $y(t) = 0$. Separate variables:

$$\frac{dy}{dt} = 3y^2$$

$$\frac{1}{y^2}\,dy = 3\,dt$$

$$\int \frac{1}{y^2}\,dy = \int 3\,dt \qquad \textbf{[6\%]}$$

$$-\frac{1}{y} = 3t + C_1 \qquad (C_1 \text{ arbitrary constant}) \qquad \textbf{[4\%]}$$

$$\frac{1}{y} = -3t + C \qquad (C \text{ arbitrary constant}) \qquad \textbf{[4\%]}$$

$$y = \frac{1}{C - 3t}$$

Use the initial condition $y(0) = 3$:

$$\frac{1}{C - 0} = 3 \quad \Longrightarrow \quad C = \frac{1}{3}. \qquad \textbf{[5\%]}$$

Hence the solution to the initial value problem is $y(t) = \dfrac{1}{1/3 - 3t}$, which may also be written in the form:

$$\boxed{y(t) = \frac{3}{1 - 9t}.} \qquad \textbf{[1\%]}$$

2. (20%) Since the intial t-value is 1, consider only $t > 0$. Divide the ODE by t:

$$y' + \frac{1}{t}y = 2. \qquad \textbf{[2\%]} \qquad (*)$$

This is *linear*. An integrating factor μ is given by

$$\mu(t) = e^{\int \frac{1}{t}\,dt} = e^{\ln t} = t. \qquad \textbf{[4\%]}$$

Multiply ODE (*) by μ to obtain (according to the theory) the equivalent ODE $(\mu y)' = \mu 2$, that is,

$$(ty)' = 2t. \qquad \textbf{[5\%]}$$

Integrate:

$$ty = \int 2t\,dt = t^2 + C \qquad (C \text{ arbitrary constant}) \qquad \textbf{[3\%]}$$

Solve for y:

$$y = t + \frac{C}{t}. \qquad \textbf{[3\%]}$$

Use the initial condition $y(1) = 2$:

$$1 + \frac{C}{1} = 2 \quad \Longrightarrow C = 1. \qquad \textbf{[2\%]}$$

Hence the solution is:

$$\boxed{y = t + \frac{1}{t}} \qquad \textbf{[1\%]}$$

See next page for #3 solution.

4. Let $f_\alpha(y) = \alpha\, y - y^2$.

   (a) **[5%]** Equilibrium points are those y for which $f_\alpha(y) = 0$. Now $f_\alpha(y) = y\,(\alpha - y)$. Hence for *every* value of α, the ODE has the equilibrium points:

   $$\boxed{y = 0, \qquad y = \alpha}$$

   (b) **[5%]** According to (a), the ODE has two equilibrium points ($y = 0$ and $y = \alpha$) when $\alpha \neq 0$, but only one equilibrium point ($y = 0$) when $\alpha = 0$. Hence one bifurcation value of the parameter α is:

   $$\boxed{\alpha = 0} \qquad \textbf{[4\%]}$$

   That there are no other bifurcation values **[1%]** will follow from the analysis of the types of equilibrium points. See (c), below.

   (c) **[10%]** We have

   $$f_\alpha{}'(y) = \alpha - 2\,y.$$

   Suppose $\alpha > 0$. At the two equilibrium points,

   $$f_\alpha{}'(0) = \alpha > 0 \implies 0 \text{ is a source}$$
   $$f_\alpha{}'(\alpha) = -\alpha < 0 \implies \alpha \text{ is a sink} \qquad \textbf{[2\%]}$$

   Suppose $\alpha < 0$. At the two equilibrium points,

   $$f_\alpha{}'(0) = \alpha < 0 \implies 0 \text{ is a sink}$$
   $$f_\alpha{}'(\alpha) = -\alpha > 0 \implies \alpha \text{ is a source} \qquad \textbf{[2\%]}$$

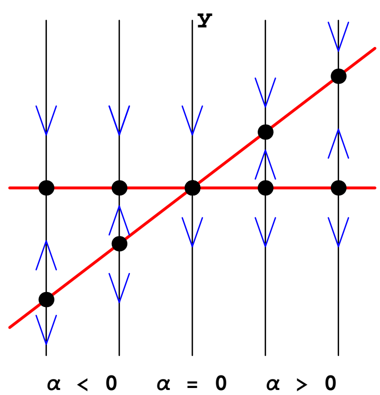
   Suppose $\alpha = 0$. [Then $f_\alpha{}'(0) = \alpha = 0$ gives **no information** about the type of equilibrium point $y = 0$ is.] In this case,

   $$f_\alpha(y) = f_0(y) = -\,y^2 \qquad \implies \qquad f_0(y) < 0 \text{ for all } y \neq 0. \qquad \textbf{[2\%]}$$

   Thus equilibrium point $y = 0$ is a node when $\alpha = 0$.

   Thus the bifurcation value $\alpha = 0$, where the number of equilibrium points changes, is also the only value where the types (sources, sinks, or nodes) of equilibrium points change. Hence there are no other bifurcation values besides $\alpha = 0$. [This completes (b).]

   When plotted in the (α, y)-plane against α, the equilibrium points lie on the *lines* $y = 0$ and $y = \alpha$. The bifurcation diagram is thus: **[4%]**



3. (20%) Let $f(t,y) = t + y^2$ and let $\Delta t = 0.1$. Calculate:

k	$\mathbf{t_k}$	$\mathbf{y_k}$	$f(t_k, y_k)$	$\Delta t\, f(t_k, y_k)$	$y_{k+1} = y_k + \Delta t\, f(t_k, y_k)$ [5%]
0	**1.0**	**2.**	5.	0.5	2.5
1	**1.1**	**2.5**	7.35	0.735	3.235
2	**1.2**	**3.235**	11.665225	1.1665225	4.4015225
3	**1.3**	**4.4015225**	—	—	—

The highlighted columns constitute the desired numerical solution. [2%]

For rounding y_3, either 0.5 or 1 points were deducted, depending on how egregious the rounding.

5. Let $f(t,y) = (t^2 + y)\,(y - 5)$.

   (a) [10%] The desired slope is:

$$\boxed{y'(2) = f\big(2, f(2)\big) = f(2,3) = -14}$$

   (b) [10%] Since $f(t,5) = 0$ for all t, then the constant function $y_1(t) = 5$ is an equilibrium solution of the ODE. [3%]
   The solution $y(t)$ with $y(0) = 2$ has initial value different from $y_1(0) = 5$. By the Uniqueness Theorem, $y(t) \neq y_1(t)$ for all t. [4%]
   Since $y(0) = 2 < 5 = y_1(0)$, then $y(t) < y_1(t)$, that is, $y(t) < 5$ for all t. [3%]

6. (20%) Let

$$\begin{aligned} t &= \text{elapsed time (min.)}, \\ W(t) &= \text{weight (lb.) of salt in tank at time } t, \\ V(t) &= \text{volume (gal.) of brine in tank at time } t. \end{aligned} \qquad [3\%]$$

Initially there is no salt in the tank: $W(0) = 0$. [2%]

We are given $V(0) = 1000/2 = 500$, [1%] and

$$V'(t) = \text{rate brine in} - \text{rate brine out} = 5 - 4 = 1. \qquad [1\%]$$

Hence

$$V(t) = t + 500. \qquad [2\%]$$

Then

$$\begin{aligned} W'(t) &= \text{rate salt in } \left(\frac{\text{lb}}{\text{min}}\right) - \text{rate salt out } \left(\frac{\text{lb}}{\text{min}}\right) \qquad [2\%] \\ &= \text{rate brine in } \left(\frac{\text{gal}}{\text{min}}\right) \cdot \text{salt concentration in } \left(\frac{\text{lb}}{\text{gal}}\right) \\ &\quad - \text{rate brine out } \left(\frac{\text{gal}}{\text{min}}\right) \cdot \text{tank's salt concentration } \left(\frac{\text{lb}}{\text{gal}}\right) \qquad [3\%] \\ &= 5 \cdot 2 - 4 \cdot \frac{W(t)}{V(t)} = 10 - \frac{4}{t+500} W(t). \qquad [5\%] \end{aligned}$$

Thus the initial value problem is:

$$\boxed{W' = 10 - \frac{4}{t+500} W, \qquad W(0) = 0} \qquad [1\%]$$