Math 300.2 **Problem Set 7 Answers** **Fall, 2007**

1. [Division Theorem for $m = 5864$ and $n = 23$.] By long division:

$$5864 = 254 \cdot 23 + 22$$

2. [Prove Lemma 2.1.7.] Just suppose there is some natural number a for which $b^m \leq a$ for every natural number m; in other words, just suppose the set A defined by

$$A = \{\, a : b^m \leq a \text{ for every } m \in \mathbb{N} \,\}$$

is nonempty. By Well-Ordering, A has a least element a. Since $a \in A$, then also $b^{m+1} \leq a$ for every natural number m, whence

$$b^m \leq \frac{a}{b}$$

for every natural number m.

(If we knew that b divided a, so that a/b were itself a natural number, then we would be done, because then a/b would be an element of A that is strictly less than the least element of A. However, there is no reason whatsoever to believe that b divides a.)

Since $b > 1$, then $a/b < a$. By the Gap Lemma, $a/b \leq a - 1$. Now $a \neq 0$ because certainly $b^0 = 1 \not\leq 0$. Thus $a \geq 1$ and so $a - 1$ is a natural number.

For each natural number m we have therefore $b^m \leq a/b \leq a - 1$. Thus $a - 1$ is an element of A. This is impossible because a is the *least* element of A. $\square$

3. [Proposition 2.2.5 part 4.] Assume $d \mid a$ and $d \mid b$. This means there exist integers m and n for which

$$a = dm, \qquad b = dn.$$

Then for arbitrary integers s and t,

$$as + bt = (dm)s + (dn)t = d(ms + nt),$$

so that $d \mid (as + bt)$, too. $\square$

4. [Exercise 2.2.9 (c).] Let m be an even integer and n be an odd integer. Then $m = 2\,s+1$ and $n = 2\,t$ for some integers s and t, respectively. We have

$$m - n = (2\,s + 1) - 2\,t = 2\,(s - t) + 1,$$

which is of the form $2\,k + 1$ for an integer k and hence is **odd**.

5. [Exercise 2.2.21 (1).] Just suppose $\sqrt{3}$ is rational. Then there exist positive integers m and n such that

$$\sqrt{3} = \frac{m}{n}. \qquad (*)$$

Without loss of generality we may assume that m and n have no common divisors other than 1 (because we may replace m and n, respectively, by their quotients when they are divided by their greatest common divisor). In particular, 3 does not divide both m and n.

Square (*) and clear of fractions to obtain

$$m^2 = 3\,n^2. \tag{**}$$

Thus *3 divides* m^2.

We claim that then *3 must divide* m. In fact, by the Division Theorem there are integers q and r with

$$m = 3 \cdot q + r, \qquad 0 \leq r < 3.$$

This means $r = 0$, 1, or 2. If $r = 1$, then

$$m^2 = (3 \cdot q + 1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1,$$

so that m^2 is not divisible by 3. And if $r = 2$, then

$$m^2 = (3 \cdot q + 2)^2 = 9q^2 + 12q + 4 = 3(3q^2 + 4q + 1) + 1,$$

so that again m^2 is not divisible by 3. Hence necessarily $r = 0$.

Since $r = 0$, then

$$m = 3 \cdot q.$$

Substitute this value in (**) to obtain

$$9\,q^2 = 3\,n^2$$

whence

$$3\,q^2 = n^2.$$

Thus n^2 is divisible by 3. By the same argument as before, n *is divisible by 3.*

Thus both m and n are divisible by 3. This is a contradiction. $\square$

Note: You could have separated from the proof above the following result (and proved it the same way it was proved above):

Lemma. Let m be an integer. If m^2 is divisible by 3, then m is divisible by 3.

6. [**Extra credit:** Exercise 2.2.21 (6).] First, note that $0 < \log_{10} 2 < 1$ (because $10^0 = 1 < 2 < 10 = 10^1$).

   Just suppose $\log_{10} 2$ is rational. Then there are integers m and n with $n \neq 0$ and

   $$\log_{10} 2 = \frac{m}{n}.$$

   Without loss of generality we may assume $n > 0$ (otherwise multiply both n and m by -1), and then $m > 0$ also (since $\log_{10} 2 > 0$). Thus

   $$0 < m < n.$$

This means that

$$10^{m/n} = 2,$$

and so

$$10^m = 2 \cdot 10^n.$$

Write 10 as $2 \cdot 5$ to obtain

$$2^m\, 5^m = 2^{n+1} 5^n,$$

and then

$$5^m = 2^{n-m+1} \cdot 5^n \tag{†}$$

Since $m < n$, then $n - m + 1 > 0$. Thus the right-hand side of (†) is even, and so the left-hand side 5^m of (†) is even, too. But 5^m is *not* even. This is a contradiction. $\square$

7. [Exercise 2.2.26 (1) (a).] The set J is **not** an ideal in $\mathbb{Z}$ because, for example, $1, 3 \in J$ but $1 + 3 = 4 \notin J$. (Another reason would be that $1 \in J$ but $2 \cdot 1 = 1 \notin J$.)

8. [Exercise 2.2.26 (4).] Let $n \in J$. We shall show that $m\,n \in J$ for every integer m.

   We use induction on m to show that $m\,n \in J$ for every nonnegative integer m. *Base step:* First, $0\,n = 0 \in J$ because $0 = n - n$ and $n \in J$. *Inductive step:* Now let m be a nonnegative integer and assume that $m\,n \in J$. Then $(m+1)\,n = m\,n + n$, and $m\,n + n \in J$ because $m\,n \in J$ (by the inductive assumption) and $n \in J$.

   Now let m be a negative integer. Then $-m$ is a positive integer, so that $(-m)\,n \in J$ by what we just proved inductively. From the base step of the induction above, $0 \in J$. Then $m\,n = 0 - (-m)\,n \in J$ because $0 \in J$ and $(-m)\,n \in J$.

   Thus J **is** an ideal in $\mathbb{Z}$. $\square$