1. (a) [Exercise 1.4.3 (8).] *Definition:* A **greatest element** of a subset A of $\mathbb{N}$ is a g for which $g \in A$ and $a \leq g$ for every $a \in A$.

   Example: Take $A = \{1, 2, 3\}$. Then 3 is a greatest element of A.

   Uniqueness of greatest element: Let $A \subset \mathbb{N}$. Assume g_1 and g_2 is each a greatest element of A. Because g_1 is a greatest element of A and $g_2 \in A$, then $g_2 \leq g_1$. Reversing the roles of g_1 and g_2, we have also $g_1 \leq g_2$. Hence $g_1 = g_2$.

   (b) [Exercise 1.4.3 (9).] *No,* a nonempty subset of $\mathbb{N}$ need not have a greatest element. In fact, $\mathbb{N}$ itself does not have a greatest element. (The reason is the same as you have seen much earlier: If $g \in \mathbb{N}$, then $g + 1 \in \mathbb{N}$ with $g + 1 > g$.)

2. [Exercise 1.4.7 (7).] Just suppose there is some integer greater than 1 that is *neither* a prime *nor* a product of primes. By the Well-Ordering Principle, there is a least such integer n_1.

   Since n_1 is not a prime, there is some integer d with $1 < d < n_1$ that divides n_1. Then there is an integer k for which

   $$n_1 = d\,k,$$

   and $1 < k < n_1$, too (because $k = 1$ would mean $d = n_1$, and $k = n_1$ would mean $d = 1$). Because d and k are integers greater than 1 but less than n_1, each is a prime or a product of primes. But then their product, n_1, is a product of primes. This is a contradiction. $\square$.

3. [Exercises 1.4.8 (5).] **Definition.** Let A be a subset of $\mathbb{Z}$. A number $b \in \mathbb{Z}$ is said to be an *upper bound of A in $\mathbb{Z}$* when $a \leq b$ for all $a \in A$. The subset A of $\mathbb{Z}$ is said to be *bounded above in $\mathbb{Z}$* when there exists some upper bound of A in $\mathbb{Z}$.

   Proposition. Each nonempty subset of $\mathbb{Z}$ that is bounded above in $\mathbb{Z}$ has a greatest element.

   Proof. Let $A \subset \mathbb{Z}$ with $A \neq \varnothing$ and assume A is bounded above in $\mathbb{Z}$. Then there exists some upper bound b of A in $\mathbb{Z}$.

   (The idea of the proof below is to reflect the set A through the origin so as to obtain a set that is bounded below in $\mathbb{Z}$; to obtain the least element of the reflected set; and then to see that the reflection of that element is the greatest element of the original set A.)

   Define $K = \{\, -a : a \in A \,\}$. The subset K of $\mathbb{Z}$ is nonempty because A is nonempty. The number $-b$ is a lower bound of K in $\mathbb{Z}$ because for each $a \in A$ we have $a \leq b$ whence $-b \leq -a$. Thus K is bounded below in $\mathbb{Z}$.

   From (4), the set K has a least element k_0. By definition of K, there is some $a_0 \in A$ for which $k_0 = -a_0$. We claim that a_0 is a greatest element of A. To see this, let $a \in A$. Now $-a \in K$, so that $k_0 \leq -a$, that is, $-a_0 \leq -a$; this implies $a \leq a_0$, as desired. $\square$

4. [Exercise 1.4.18 (3).] Use *strong* induction on n.

Base step: ($n = 1$). First, $F_1 = 1 = \left(\frac{13}{8}\right)^0 = \left(\frac{13}{8}\right)^{1-1}$.

Inductive step. Now let $n \geq 1$ and assume that, for *every* positive integer $k \leq n$,

$$F_k \leq \left(\frac{13}{8}\right)^{k-1} \qquad \text{and, if } k > 1, \text{ then } F_k < \left(\frac{13}{8}\right)^{k-1}.$$

What must be deduced is that $F_{n+1} \leq (13/8)^n$ and, if $n+1 > 1$, then $F_{n+1} < (13/8)^n$. Of course $n+1 > 1$ because $n \geq 1$, so all that must be deduced is that $F_{n+1} < (13/8)^n$.

By the recursive definition of the Fibonacci numbers, $F_{n+1} = F_n + F_{n-1}$. To apply the inductive assumption to both n and $n-1$ will require not just that $n \leq n$ and $n - 1 \leq n$ (both of which are certainly true), but also that $n \geq 1$ (which is true) and that $n - 1 \geq 1$—which is *not* so unless $n \geq 2$. Hence we must treat separately the case $n = 2$.

In the case $n = 2$: $F_{2+1} = F_3 = 2 < \dfrac{169}{64} = \left(\dfrac{13}{8}\right)^2 = \left(\dfrac{13}{8}\right)^{(2+1)-1}$

Now suppose $n \geq 2$, so that $1 \leq n \leq n$ and $1 \leq n - 1 \leq n$. By the inductive assumption,

$$F_n \leq \left(\frac{13}{8}\right)^{n-1}, \qquad F_{n-1} \leq \left(\frac{13}{8}\right)^{n-2}.$$

Then:

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &\leq \left(\frac{13}{8}\right)^{n-1} + \left(\frac{13}{8}\right)^{n-2} \\ &= \left(\frac{13}{8}\right)^{n-2} \left(\frac{13}{8} + 1\right) = \left(\frac{13}{8}\right)^{n-2} \frac{21}{8}. \end{aligned}$$

In order to complete the deduction that $F_{n+1} < (13/8)^n$, it remains only to show that

$$\left(\frac{13}{8}\right)^{n-2} \frac{21}{8} < \left(\frac{13}{8}\right)^n.$$

The latter inequality is equivalent to

$$\left(\frac{13}{8}\right)^{-2} \frac{21}{8} < 1$$

that is,

$$\left(\frac{8}{13}\right)^2 \frac{21}{8} < 1.$$

But

$$\left(\frac{8}{13}\right)^2 \frac{21}{8} = \frac{168}{169} < 1,$$

as needed. (Whew—rather close!) □

5. [Prove Proposition 1.5.6.] Let n and j be integers with $0 \le j \le n$. If $j = 0$, then $\binom{n}{j-1} = 0$ by definition, so that

$$\binom{n+1}{j} = \binom{n+1}{0} = 1 = 0 + 1 = \binom{n}{j-1} + \binom{n}{0} = \binom{n}{j-1} + \binom{n}{j}.$$

Now suppose $j > 0$, so that $1 \le j \le n$. By definitions of the coefficients here,

$$\binom{n+1}{j}$$

$$\begin{aligned}
\binom{n}{j-1} + \binom{n}{j} &= \frac{n!}{(j-1)! \cdot (n - [j-1])!} + \frac{n!}{j! \cdot (n-j)!} \\
&= \frac{n!}{(j-1)! \cdot (n-j+1)!} + \frac{n!}{j! \cdot (n-j)!} \\
&= \frac{j \cdot n!}{j \cdot (j-1)! \cdot (n-j+1)!} + \frac{(n-j+1) \cdot n!}{j! \cdot (n-j)! \cdot (n-j+1)} \\
&= \frac{j \cdot n!}{j! \cdot (n-j+1)!} + \frac{(n-j+1) \cdot n!}{j! \cdot (n-j+1)!} \\
&= \frac{j \cdot n! + (n-j+1) \cdot n!}{j! \cdot (n-j+1)!} = \frac{\big(j + (n-j+1)\big)\, n!}{j! \cdot (n-j+1)!} \\
&= \frac{(n+1) \cdot n!}{j! \cdot (n-j+1)!} \\
&= \frac{(n+1)!}{j! \cdot (n+1-j)!} = \binom{n+1}{j}. \quad \square
\end{aligned}$$