Math 300.2 Problem Set 5 Answers Fall, 2007

1. [Exercise 1.2.11 (b).] *Base step:* ($n = 0$). First, if we take $m = 0$ in the definition of addition, we obtain $0 + 0 = 0$.

   Inductive step. Now let $n \geq 0$ and assume that $0 + n = n$. Then:

$$\begin{aligned} 0 + (n+1) &= (0+n)+1 && \text{(by the recursive definition of addition)} \\ &= n+1 && \text{(by the inductive assumption)} \end{aligned}$$

2. [Exercise 1.2.11 (c).] At the point in the inductive step, marked (*) below, you will need the following result—a special case!—which was proved in class:

   Lemma. For all nonnegative integers n, we have $1 + n = n + 1$.

   Proof of main result.

   Note: You may use induction on either m or on n. Below we use induction on m. You need to choose which of two (equivalent) statements you are trying to prove:

   - $(\forall n \in \mathbb{N})(\forall m \in \mathbb{Z})(m + n = n + m)$; or else
   - $(\forall m \in \mathbb{N})(\forall n \in \mathbb{N})(m + n = n + m)$.

   The way you write the proof will depend on which one you choose. Below is the proof where we choose the first of the two statements to prove. If, however, you choose the second statement, then the base step will be that, for *every* integer n, the sum $0 + n = n + 0$. And in this case in the inductive step, when you let $m \geq 0$, the inductive assumption will be that, for *every* integer n, the sum $m + n = n + m$; and what you must deduce is that, for *every* integer n, the sum $(m + 1) + n = n + (m + 1)$.

   The advantage of proving the first of the two statements is that you may fix n once at the very start of the proof. If, however, you are proving the second statement, then you must repeatedly say things about *every* n.

   Fix $n \in \mathbb{N}$. We shall use induction on m to prove that, for each $m \in \mathbb{N}$, we have $m + n = n + m$.

   Base step: ($m = 0$). From 1.2.11 (b), we have $0 + n = n = n + 0$ for every integer $n \geq 0$.

   Inductive step. Now let $m \geq 0$ and assume that

$$m + n = n + m.$$

   We wish to deduce that $(m + 1) + n = n + (m + 1)$.

   We have

$$\begin{aligned} (m+1)+n &= m+(1+n) && \text{(by associativity of addition)} \\ &= m+(n+1) && \text{(by the preceding lemma)} \qquad (*) \\ &= (m+n)+1 && \text{(by the recursive definition of addition)} \\ &= (n+m)+1 && \text{(by the inductive assumption)} \\ &= n+(m+1) && \text{(by the recursive definition of addition),} \end{aligned}$$

   as desired. $\square$

   Note: Instead of "by the recursive definition of addition", you could equally well have said "by associativity of addition".

3. [Exercise 1.2.14 (1).] Take $X = \mathbb{R}$, $z = 1$, and $G\colon X \to X$ to be the function defined by:

$$G(x) = a \cdot x \qquad (x \in X)$$

4. [Exercise 1.2.11 (2).] Fix natural numbers m and n with $m < n$. This means there exists some positive integer d for which

$$m + d = n.$$

Now let k be a natural number. Then

$$\begin{aligned} (m + k) + d &= (m + d) + k && \text{(associative \& commutative laws)} \\ &= n + k. \end{aligned}$$

According to the definition of $<$, this means that $m + k < n + k$. $\square$

Note: You could also use induction on k, although it would be a waste of effort since all the induction needed was already done in establishing the commutative and associative laws of addition!

5. [Exercise 1.2.22 (2).] We wish to show that, for every integer $n \geq 1$, the Fibonacci numbers F_n and F_{n+1} are *not* both divisible by any integer except 1. We use induction on n.

Base step: $(n = 1)$. The first two Fibonacci numbers $F_1 = 1$ and $F_2 = 1$ are certainly not divisible by any integer except 1.

Inductive step. Let $n \geq 1$ and assume that F_n and F_{n+1} are *not* both divisible by any integer except 1. We wish to deduce that F_{n+1} and $F_{(n+1)+1} = F_{n+2}$ are not divisible by any integer except 1.

Just suppose, to the contrary, there is some integer d for which both F_{n+1} and F_{n+2} *are* divisible by d. This means there are integers s and t for which

$$F_{n+1} = d \cdot s, \qquad F_{n+2} = d \cdot t.$$

Note that $n + 2 \geq 3$. Then by the recursive relation for Fibonacci numbers,

$$F_{n+2} = F_n + F_{n+1}$$

so that

$$F_{n+2} = F_n + d \cdot s.$$

Now $F_{n+2} = d \cdot t$, so from the preceding equalities, $d \cdot t = F_n + d \cdot s$, whence

$$F_n = d \cdot (t - s).$$

This means that F_n is divisible by d. However, by our supposition F_{n+1} is also divisible by d. This contradicts the inductive assumption. $\square$

6. [Exercise 1.3.4 (4).] We shall use the fact, previously stated in class (without proof), that a natural number m is odd if and only if it has the form $m = 2k + 1$ for some natural number k. Another way to say this is that m is odd if and only if $m = 2(j - 1) + 1 = 2j - 1$ for some *positive* integer j. (You may work with either form.) Thus the first n natural numbers are $1 = 2 \cdot 1 - 1,\ 3 = 2 \cdot 2 - 1, \ldots,\ 2 \cdot n - 1$. And so the sum we want is $\sum_{j=1}^{n} (2j - 1)$.

   By Proposition 1.3.2,

$$\begin{aligned}\sum_{j=1}^{n} (2j-1) &= \sum_{j=1}^{n} \big[(2j + (-1)1\big] = \sum_{j=1}^{n} 2j + \sum_{j=1}^{n} (-1)1 \\ &= 2\sum_{j=1}^{n} j + (-1)\sum_{j=1}^{n} 1 = 2\sum_{j=1}^{n} j - \sum_{j=1}^{n} 1\end{aligned}$$

   We know $\sum_{j=1}^{n} j = n(n-1)/2$. And "obviously" $\sum_{j=1}^{n} 1 = n$ because

$$\sum_{j=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n$$

   (strictly speaking, this needs proof—by induction, of course; see below).

   Hence

$$\sum_{j=1}^{n} (2j-1) = 2 \cdot \frac{n(n+1)}{2} - n = n^2 + n - n = n^2,$$

   and so our formula is:

$$\boxed{\sum_{j=1}^{n} (2j-1) = n^2}$$

   (For only 60% credit, you could guess this formula from examining several values of n and then use induction to prove it.)

   Optional: **Proof by induction that** $\sum_{j=1}^{n} 1 = n$ **for each** $n = 1, 2, \ldots$**.** First, from the initial condition in the recursive definition of summation, $\sum_{j=1}^{1} 1 = s$. Now let n be a positive integer and assume $\sum_{j=1}^{n} 1 = n$. From the recurrence relation in the recursive definition of summation,

$$\sum_{j=1}^{n+1} 1 = \left(\sum_{j=1}^{n} 1\right) + 1 = n + 1$$

   (where in the final equality we used the inductive assumption). $\square$

7. (a) [Exercise 1.3.6 (3).] For help, you might also try factoring $x^5 - y^5$ so as to obtain

$$x^5 - y^5 = (x - y)\,(x^4 + x^3y + x^2y^2 + xy^3 + y^4).$$

The generalization is that, for all integers $n \geq 2$,

$$x^n - y^n = (x-y)\sum_{j=0}^{n-1} x^{n-j-1}y^j \tag{*}$$

We prove this by using properties of summation:

$$\begin{aligned}
(x-y)\sum_{j=0}^{n-1} x^{n-j-1}y^j &= x\sum_{j=0}^{n-1} x^{n-j-1}y^j - y\sum_{j=0}^{n-1} x^{n-j-1}y^j \\
&= \sum_{j=0}^{n-1} x^{n-j}y^j - \sum_{j=0}^{n-1} x^{n-j-1}y^{j+1} \\
&= \sum_{j=0}^{n-1} x^{n-j}y^j - \sum_{j=0+1}^{n-1+1} x^{n-(j-1)-1}y^{(j-1)+1} && \text{(shift index in 2nd sum)} \\
&= \sum_{j=0}^{n-1} x^{n-j}y^j - \sum_{j=1}^{n} x^{n-j}y^j \\
&= x^n y^0 + \sum_{j=1}^{n-1} x^{n-j}y^j - \sum_{j=1}^{n-1} x^{n-j}y^j - x^0 y^n && \text{(peel off terms)} \\
&= x^n - y^n && \text{(cancel the two sums)}
\end{aligned}$$

(b) [Exercise 1.3.7 (a).] First, assume a is a root of $p(x)$. Then $p(a) = 0$ so that:

$$\begin{aligned}
p(x) &= p(x) - p(a) \\
&= \sum_{j=0}^{n} c_j x^j - \sum_{j=0}^{n} c_j a^j \\
&= \sum_{j=0}^{n} c_j (x^j - a^j) \\
&= (x^0 - a^0) + (x^1 - a^1) + \sum_{j=2}^{n} c_j(x^j - a^j) = 0 + (x-a) + \sum_{j=2}^{n} c_j(x^j - a^j)
\end{aligned}$$

Now by part (a) of this problem, for each $j = 2, 3, \ldots, n$ the factor $(x^j - a^j)$ in the sum is divisible by $x - a$. It follows that the entire quantity on the right in last line above is divisible by $x - a$. In other words, $p(x)$ is divisible by $x - a$. Conversely, suppose $p(x)$ is divisible by $x - a$. Then there is a polynomial $q(x)$ with integer coefficients for which

$$p(x) = (x-a)q(x).$$

Substitute a for x here to obtain

$$p(a) = (a-a)q(a) = 0 \cdot q(a) = 0,$$

which means that a is a root of $p(x)$. $\square$

8. [$m \not< m$ for all $m \in \mathbb{N}$.] We use induction on m.

Base step: ($n = 0$). Just suppose $0 < 0$. Then according to the definition of $<$ there exists some $k \in \mathbb{N}^*$ for which $0 + k = 0$. By the commutativity of addition, $k + 0 = 0$. However, according to the definition of addition, $k + 0 = k$. Thus $k = 0$, which is impossible because $k \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

Inductive step. Let $m \in \mathbb{N}$ and assume $m \not< m$. Just suppose $m + 1 < m + 1$. Then according to the definition of $<$ there exists some $k \in \mathbb{N}^*$ for which

$$(m + 1) + k = m + 1.$$

By the associative and commutative laws for addition,

$$(m + k) + 1 = m + 1.$$

Because the successor function $\sigma \colon \mathbb{N} \to \mathbb{N}$ is injective,

$$m + k = m.$$

By the definition of $<$, this means $m < m$, contrary to the inductive assumption. $\square$