Math 300.2 **Problem Set 4 Answers** **Fall, 2007**

1. [Exercise 1.1.26 (e).] Fix a real number $x \geq 0$. We use induction to show that $(1+x)^n \geq 1+nx$ for all $n = 0, 1, 2, \ldots$.

   Base step ($n = 0$): First, $(1+x)^0 = 1 \geq 1 + 0 = 1 + 0 \cdot x$. [2%]

   Inductive step. Let n be a nonnegative integer and assume that $(1+x)^n \geq 1+nx$. [1%] We wish to deduce that $(1+x)^{n+1} \geq 1+(n+1)x$. [1%] Then

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)\cdot(1+x)^n && \text{[2\%]}\\
&\geq (1+x)\cdot(1+nx) && \text{[2\%] (by the inductive assumption)}\\
&= 1+x+nx+nx^2 = (n+1)x+1+nx^2 && \text{[1\%]}\\
&\geq (n+1)x && \text{[1\%]} \quad \square
\end{aligned}$$

2. [Exercise 1.1.26 (i).] Calculations reveal that, although the inequality is true for $1 \leq n \leq 4$, it is false for $5 \leq n \leq 15$. However, it appears to be true again for $\boxed{n \geq 16}$. [3%] We shall prove that this is the case.

   Discussion. Since $\log_2 n \leq \sqrt{n} \iff n \leq 2^{\sqrt{n}}$, you could prove, instead, the equivalent equality $n \leq 2^{\sqrt{n}}$ for $n = 16, 17, 18, \ldots$.

   I first tried that by induction on n. In the inductive step, after assuming that $n \leq 2^{\sqrt{n}}$, I tried to deduce that $n+1 \leq 2^{\sqrt{n+1}}$ by proving that $2^{\sqrt{n}}+1 \leq 2^{\sqrt{n+1}}$ (which clearly would suffice). However, to prove the latter inequality I would have to directly compare the two functions of the variable n without any further use of the inductive assumption. So I abandoned the attempt at induction and began anew.

   (Legitimate proofs by induction on n are welcome!)

   Proof. [7%] We prove that $\log_2 x \leq \sqrt{x}$, for all *real* $x \geq 16$. We have $\log_2 x = (\log_e x)(\log_e 2) = (\ln x)(\ln 2)$. Then what we want to prove is,

$$\frac{\ln x}{\ln 2} \leq \sqrt{x} \qquad (x \geq 16),$$

   or, equivalently:

$$\ln x \leq (\ln 2)\sqrt{x} \qquad (x \geq 16)$$

   Let $f(x) = \ln x$ and $g(x) = (\ln 2)\sqrt{x}$. We shall show that $f(16) = g(16)$ and that $f'(x) \leq g'(x)$ for $x \geq 16$, and this will imply that $f(x) \leq g(x)$ for $x \geq 16$, as desired.

   First,

$$f(16) = \ln 16 = \ln(2^4) = 4(\ln 2) = (\ln 2)\sqrt{16} = g(16).$$

   Next, $f'(x) = 1/x$ and $g'(x) = (\ln 2)/(2\sqrt{x})$, and

$$f'(x) = \frac{1}{x} \leq \frac{\ln 2}{2\sqrt{x}} = g'(x) \iff \frac{2}{\ln 2} \leq \sqrt{x}.$$

   Now the latter inequality is certainly true when $x \geq 16$. Indeed, if $x \geq 16$, then $\sqrt{x} \geq 4$, and $4 \geq 2/\ln 2$ because, equivalently, $\ln 2 > 1/2$. $\square$

3. [Exercise 1.1.27 (b).] Denote by H_n the number of handshakes for n people.

Discovery. (This part **[4%]**.) For just 1 person, there are no handshakes, so let's restrict our consideration to at least 2 people. For 2 people, there is exactly 1 handshake, between those two. For 3 people—call them A, B, and C—there are 3 handshakes, namely: A with B, A with C, and B with C. For 4 people—call them A, B, C, and D—there are 6 handshakes, namely: A with B, A with C, A with D, B with C, B with D, C with D. You could try, say, 5 people as well, but look at the numbers already:

n	2	3	4
H_n	1	3	6

Probably no pattern is yet evident in these numbers. So try another approach....

Suppose there are n people; $P_1, P_2, \dots, P_n$. Then P_1 shakes hand with each of the $n-1$ people $P_2, P_3, \dots, P_n$. And aside from shaking hands with P_1 (which we already counted), P_2 shakes hands with the $n-2$ people $P_3, P_4, \dots, P_n$. Similarly, aside from shaking hands with P_1 and P_2 (which we already counted), P_3 shakes hands with the $n-3$ people $P_4, P_5, \dots, P_n$. Etc. Finally, aside from shaking hands with $P_1, P_2, \dots, P_{n-2}$ (which we already counted for those folks), P_{n-1} shakes hands with just 1 person, namely, P_n. And we have now already counted all the handshakes with P_n. So the total number H_n of handshakes is given by:

$$H_n = (n-1) + (n-2) + (n-3) + \cdots + 1 = \sum_{j=1}^{n-1} j \qquad \textbf{[2.5\%]}$$

To find a closed form of the formula for H_n, use the formula from Example 1.1.15 for the sum of the first n positive integers (replacing n there by $n-1$) so as to obtain::

$$\boxed{H_n = \frac{n(n-1)}{2} \qquad (n = 2, 3, \dots)} \qquad \textbf{[1.5\%]} \tag{*}$$

Because of that 'Etc." in our counting, we need an inductive proof of (*).

Proof. (This part **[6%]**.) We prove (*) by induction on n.

Base step ($n = 2$): For $n = 2$ people, clearly there is just one handshake, between those 2 people. Thus $H_2 = 1 = 2(2-1)/2$, as required. **[1%]**

Inductive step: Now let $n \geq 2$ and assume $H_n = n(n-1)/2$. **[1%]** We wish to deduce that $H_{n+1} = (n+1)\big((n+1)-1\big)/2$, that is, $H_{n+1} = (n+1)n/2$. **[1%]**

Denote the $n+1$ people by $P_1, P_2, \dots, P_n, P_{n+1}$. A handshake between two of these people is either a handshake between some two of the first n people $P_1, P_2, \dots, P_n$ or else a handshake between P_{n+1} and one of those n people. Thus

$$H_{n+1} = H_n + K_n \qquad \textbf{[1\%]}$$

where K_n is the number of handshakes between P_{n+1} and one of those n other people. Clearly $K_n = n$. **[1%]** And by the inductive assumption, $H_n = n(n-1)/2$. Hence:

$$\begin{aligned} H_{n+1} &= \frac{n(n-1)}{2} + n \\ &= \frac{n(n-1) + 2n}{2} = \frac{n\big((n-1)+2\big)}{2} = \frac{(n+1)n}{2} \qquad \square \qquad \textbf{[1\%]} \end{aligned}$$

Note: We regard a handshake as taking place between two people, without consideration of one person first extending the handshake to the other. In other words, a handshake between two people is represented by the *set* of those two people. So in more mathematical terms, the problem was to count the *number of 2-element subsets of a set of n elements.* We discovered, and proved, that this number is $n(n-1)/2$.

Another way to represent this number is as the *binomial coefficient* $\binom{n}{2}$. As we shall later see,

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)(n-2)(n-3)\cdots 2\cdot 1}{2\cdot 1\cdot (n-2)(n-3)\cdots 2\cdot 1} = \frac{n(n-1)}{2}.$$

[Of course this formula also requires proof, especially in view of the dots $(\cdots)$ there.]

4. [Exercise 1.1.27 (d).] Denote by S_n the number of subsets of an n-element set.

Discovery. (This part **[3%]**.) Through examples we've already observed at least the following:

n	0	1	2	3
S_n	1	2	4	8

The pattern here , we conjecture, is:

$$\boxed{S_n = 2^n \qquad (n = 0, 1, 2, \ldots)}$$

Here's another way of arriving at this guess. Number as $x_1, x_2, \ldots, x_n$ the elements of an n-element set X. An arbitrary subset A of X is determined by which of those n elements belong to it: either $x_1 \in A$ or else $x_1 \notin A$; either $x_2 \in A$ or else $x_2 \notin A$, $\ldots$, either $x_n \in A$ or else $x_n \notin A$. For each of the n elements x_i of X, there is a choice from two mutually exclusive alternatives to be made, namely, whether $x_i \in A$ or else $x_i \notin A$. Thus there are exactly

$$\underbrace{2\cdot 2\cdot 2\cdot \cdots \cdot 2}_{n} = 2^n$$

choices in all.

Proof. (This part **[7%]**.) More precisely, we use induction on n to prove:

Every n-element set has exactly 2^n subsets. $\qquad (n = 0, 1, 2, \ldots)$

Base step ($n = 0$): A set with 0 elements is just the empty set $\varnothing$, and this has only 1 subset, namely, itself. Thus $S_0 = 1$, as required. **[1%]**

Inductive step: Now let $n \geq 0$ and assume $S_n = 2^n$, that is—and this is essential to say—***every*** *n-element set has exactly 2^n subsets.* **[1%]**

Let X be an arbitrary $(n+1)$-element set. We wish to deduce that X has exactly 2^{n+1} subsets. **[1%]**

Number the elements of X as $x_1, x_2, \ldots, x_n, x_{n+1}$. Now x_{n+1} is an element of some subsets of X and not an element of others. In other words, subsets A of X are of two distinct kinds: those $A \subset X$ for which $x_{n+1} \notin A$ and those $A \subset X$ for which $x_n \in A$.

Then the total number S_{n+1} is the sum of the numbers of each of these two kinds of subsets. **[1%]**

Count the second kind of subsets. A subset A of X with $x_{n+1} \notin A$ is just a subset of $X \setminus \{x_{n+1}\} = \{x_1, x_2, \ldots x_n\}$. Now $X \setminus \{x_{n+1}\}$ is an n-element set, and by the inductive assumption this n-element set has exactly 2^n subsets. Thus there are exactly 2^n subsets A of X for which $x_{n+1} \notin A$. **[1%]**

Count the first kind of subsets. Think of each subset A of X with $x_{n+1} \in A$ as: remove the element x_{n+1} from A so as to obtain a subset B of $X \setminus \{x_{n+1}\}$, and then put x_{n+1} back into A to obtain $A = B \cup \{x_{n+1}\}$. In other words: If $A \subset X$ with $x_{n+1} \in A$, then

$$A = B \cup \{x_{n+1}\}$$

with

$$B = A \setminus \{x_{n+1}\} \subset X \setminus \{x_{n+1}\};$$

and conversely, if $B \subset X \setminus \{x_{n+1}\}$, then

$$A = B \cup \{x_{n+1}\} \subset X.$$

is a subset of X with $x_{n+1} \in A$. Thus there are exactly as many subsets A of X with $x_{n+1} \in A$ as there are subsets B of $X \setminus \{x_{n+1}\}$. Now $X \setminus \{x_{n+1}\}$ is an n-element set, so by the inductive assumption $X \setminus \{x_{n+1}\}$ has exactly 2^n subsets. Hence the number of subsets A of X with $x_{n+1} \in A$ is also 2^n. **[1%]**

There are thus 2^n subsets of X of the first kind and 2^n subsets of the second kind. Hence the total number of subsets of X is

$$2^n + 2^n = 2 \cdot 2^n = 2^{n+1}, \qquad \textbf{[1\%]}$$

as desired. $\square$