Math 300.2 **Problem Set 3 Answers** **Fall, 2007**

1. [Exercise B.1.9 (2) (g).] Let $A \subset X$. To see that $A \subset f^{-1}\big(f(A)\big)$, let $a \in A$. Then $f(a) \in f(A)$. Then there exists some $b \in f(A)$ for which $f(a) = b$, namely, $b = f(a)$. By definition of inverse image, $a \in f^{-1}\big(f(A)\big)$. $\square$

   We show by counterexample that equality does *not* need to hold. Take, for example, $X = Y = \mathbb{R}$, $f\colon X \to Y$ the function given by $f(x) = x^2$ for all $x \in X$, and $A = \{\, x \in \mathbb{R} : x \leq 0 \,\}$, the set of nonpositive reals. Clearly $f(A) = \mathbb{R}^+$, the set of nonnegative reals. Now $f^{-1}(\mathbb{R}^+) = \mathbb{R}$ [since also $f(x) \in \mathbb{R}^+$ for every $x > 0$]. Thus $A \neq f^{-1}\big(f(A)\big)$.

   [*Note:* Why would suspect that equality need not hold? Look at what happens if you try to prove that it does: Let $x \in f^{-1}\big(f(A)\big)$. This means $f(x) \in f(A)$, so $f(x) = f(a)$ for some $a \in A$. However, we *cannot* deduce from that that $x = a$, and so cannot deduce that then $x \in A$. Indeed, this suggests looking for an example where the function f is not injective. The squaring function with domain $\mathbb{R}$ is one such.]

2. [Exercise B.2.6 (2).] We must establish both existence and uniqueness of h.

   Uniqueness. Let $h\colon Z \to X \times Y$ be a map making the diagram commutative. Let $z \in Z$. Since $h(z) \in X \times Y$, write

   $$h(z) = (x, y)$$

   where $x \in X$ and $y \in Y$. Then

   $$x = p\big(h(z)\big) \quad \text{and} \quad y = q\big(h(z)\big).$$

   According to the commutativity of the diagram,

   $$p\big(h(z)\big) = f(z) \quad \text{and} \quad q\big(h(z)\big) = g(z).$$

   Hence

   $$x = f(z) \quad \text{and} \quad y = g(z).$$

   Thus

   $$h(z) = \big(f(z), g(z)\big). \tag{*}$$

   This establishes uniqueness.

   Existence. Now define the rule for $h\colon Z \to X \times Y$ by the formula (*). [This does define a function from Z to $X \times Y$ because for each $z \in Z$, we have $f(z) \in X$ and $f(z) \in Y$ so that $\big(f(z), g(z)\big) \in X \times Y$.] It remains to check that this h makes the diagram commutative. For each $z \in Z$:

   $$p\big(h(z)\big) = p\big(f(z), g(z)\big) = f(z), \quad q\big(h(z)\big) = q\big(f(z), g(z)\big) = g(z),$$

   so that $p \circ h = f$ and $q \circ h = g$, as desired. $\square$

3. [Exercise B.3.7.] First we show f is injective. Let $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$. Then $g\big(f(x_1)\big) = g\big(f(x_2)\big)$, that is, $(g \circ f)(x_1) = (g \circ f)(x_2)$. since $g \circ f$ is injective, it follows that $x_1 = x_2$.

   Second we show g is surjective. Let z Z. Since $g \circ f$ is surjective, there is some $x \in X$ for which $(g \circ f)(x) = z$. Define $y = f(x)$. Then $y \in Y$ with $g(y) = g\big(f(x)\big) = z$. $\square$

4. [Exercise B.3.9 (2).] Let $f\colon X \to Y$ and $g\colon Y \to Z$ be bijections. From Prop. B.3.6, the composite $g \circ f\colon X \to Z$ is also bijective.

   Now both $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are maps $Z \to X$. To see they have the same values, let $z \in Z$. Define

   $$y = g^{-1}(z), \qquad x = f^{-1}(y).$$

   so that

   $$g(y) = z, \qquad f(x) = y.$$

   Then

   $$(g \circ f)(x) = g\big(f(x)\big) = g(y) = z,$$

   so that

   $$(g \circ f)^{-1}(z) = x.$$

   However,

   $$\big(f^{-1} \circ g^{-1}\big)(z) = f^{-1}\big(g^{-1}(z)\big) = f^{-1}(y) = x$$

   by definition of x and y. Thus

   $$\big(f^{-1} \circ g^{-1}\big)(z) = \big(f^{-1} \circ g^{-1}\big)(z),$$

   as desired. $\square$

   Note: Another way to show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ is to use Prop. B.3.12.

5. [Exercise B.3.10.]

   (a) If $\alpha = 0$, then $f(x) = \beta$ is constant.

   Conversely, suppose f is constant. Now $f(0) = \beta$, so that $f(x) = \beta$ for *all* $x \in \mathbb{R}$. In particular, $f(1) = \beta$. But $f(1) = \alpha(1) + \beta$, so that $\alpha + \beta = \beta$ and consequently $\alpha = 0$.

   Hence f is constant if and only if $\alpha = 0$.

   (b) If $\alpha = 0$, certainly f cannot be injective (because it would be constant).

   Conversely, suppose $\alpha \neq 0$. Then for all $x_1, x_2 \in R$, if $f(x_1) = f(x_2)$, then $\alpha x_1 + \beta = \alpha x_2 + \beta$ whence $x_1 = x_2$ (because $\alpha \neq 0$). Thus f is injective.

   Hence f is injective if and only if $\alpha \neq 0$.

   (c) If $\alpha = 0$, certainly f cannot be surjective (because it would take the constant value β at all $x \in \mathbb{R}$).

   Conversely, suppose $\alpha \neq 0$. Let $y \in \mathbb{R}$. By solving the equation $\alpha x + \beta = y$ for x to obtain $x = (y - \beta)/\alpha$ we see that $f(x) = y$ for this x. Thus f is surjective.

   Hence f is surjective if and only if $\alpha \neq 0$.

   (d) From (b) and (c), f is bijective if and only if $\alpha \neq 0$.