Math 300.2 **Problem Set 1 Answers** **Fall, 2007**

1. (a) [Exercise L.4.6 for part 8 of Tautology L4.5.] Your intermediate columns might be in a different order from mine.

P	Q	R	$Q\&R$	$P \vee (Q\&R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \& (P \vee R)$	$P \vee (Q\&R) \iff (P \vee Q) \& (P \vee R)$
T	T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F	T
F	F	T	F	F	F	T	F	T
F	F	F	F	F	F	F	F	T

(b) [Exercise L.4.11 for part 9 of Tautology L.4.5.] Begin by examining the negation of the desired formula $P \,\&\, (Q \vee R)$:

$$
\begin{aligned}
\neg\big(P \,\&\, (Q \vee R)\big) &\iff \neg P \vee \neg(Q \vee R) && (1)\\
&\iff \neg P \vee (\neg Q \,\&\, \neg R) && (2)\\
&\iff (\neg P \vee \neg Q) \,\&\, (\neg P \vee \neg R) && (3)\\
&\iff \big(\neg(P \,\&\, Q)\big) \,\&\, \big(\neg(P \,\&\, R)\big) && (4)\\
&\iff \neg\big((P \,\&\, Q) \vee (P \,\&\, R)\big) && (5)
\end{aligned}
$$

[Here steps (1) and (2) each follows by applying a De Morgan Law; (3) follows from Tautology L.4.5 part 8; (4) follows by using a De Morgan Law twice; and (5) applies by using a De Morgan Law.]

So far, $\neg\big(P \,\&\, (Q \vee R)\big) \iff \neg\big((P \,\&\, Q) \vee (P \,\&\, R)\big)$. By double negation, taking the negation of both sides of this equivalence yields the desired result.

2. (a) [Prove Tautology L.4.8 with a truth table.] Your intermediate columns might be in a different order from mine.

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$	$(P \Rightarrow Q) \iff (\neg P \vee Q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

(b) [Prove Tautology L.4.10, part 2.]

$$
\begin{aligned}
\neg(P \,\&\, Q) &\iff \neg\big(\neg(\neg P) \,\&\, \neg(\neg Q)\big) && (6)\\
&\iff \neg\,\big(\neg\big((\neg P) \vee (\neg Q)\big)\big) && (7)\\
&\iff (\neg P) \vee (\neg Q) && (8)
\end{aligned}
$$

[Here step (6) uses double negation twice; (7) uses the first part of Tautology L.4.10; and (8) again uses double negation.]

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3. [Exercise A.1.2 (2).] *Method 1:* use the definition of power set directly to show that $X \subset Y$ and $Y \subset X$. Let X and Y be sets and assume $\mathcal{P}(X) = \mathcal{P}(Y)$. Since $X \in \mathcal{P}(X)$, then also $X \in \mathcal{P}(Y)$. This means $X \subset Y$. Similarly (by reversing the roles of X and Y, we see that $Y \subset X$. Hence $X = Y$.

   Method 2: work with elements of X and Y to show $x \in X \iff x \in Y$. Observe that, for a set Z and an element z, we have:

$$z \in Z \iff \{z\} \subset Z \iff \{z\} \in \mathcal{P}(Z)$$

   Let X and Y be sets and assume $\mathcal{P}(X) = \mathcal{P}(Y)$. We show, first, that $X \subset Y$. Let $x \in X$. Then $\{x\} \subset X$, that is, $\{x\} \in \mathcal{P}(X)$. By the assumption, also $\{x\} \in \mathcal{P}(Y)$, that is, $\{x\} \subset Y$. This means $x \in Y$.

   By exactly the same argument but with the roles of X and Y reversed, $Y \subset X$. Hence $X = Y$. $\square$

4. [Exercise A.1.3 (d), 2nd part] Let A and B be any sets.

   First we show that $A \cap B = A \implies A \subset B$. Assume $A \cap B = A$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$, so in particular $x \in B$. Thus $A \subset B$.

   Second we show, conversely, that $A \subset B \implies A \cap B = A$. Assume $A \subset B$. Now $A \cap B \subset A$ no matter what the sets A and B are. To see the reverse inclusion, let $x \in A$. Since $A \subset B$, then also $x \in B$, and so $x \in A \cap B$. Thus $A \cap B = A$. $\square$

$$A \cap B = A \iff A \subset B.$$

5. [Prove Prop. A.1.4, 2nd part, about $X \setminus (A \cap B)$.] For all x:

$$\begin{aligned} x \in X \setminus (A \cap B) &\iff x \in X \;\&\; x \notin A \cap B \\ &\iff x \in X \;\&\; (x \notin A \text{ or } x \notin B) \\ &\iff (x \in X \;\&\; x \notin A) \text{ or } (x \in X \;\&\; x \notin B) \\ &\iff x \in X \setminus A \text{ or } x \in X \setminus B \\ &\iff x \in (X \setminus A) \cup (X \setminus B) \qquad \square \end{aligned}$$

   Note: It is tempting to do here something analogous to what was done (taking negations) in the solution to # 2b. In other words, to start with $X \setminus \bigl(X \setminus (A \cap B)\bigr)$ and to use the first of De Morgan's Laws (Prop. A.1.4 part 1). This approach would work in the case that $A \subset X$ and $B \subset X$. Unfortunately, for a set X and an arbitrary set S, it is *not* necessarily true that $X \setminus (X \setminus S) = S$; in fact, $X \setminus (X \setminus S) = X \cap S$, so that $X \setminus (X \setminus S) \neq S$ when $S \not\subset X$.

6. [Exercise A.1.5 (4).] Let A and B be subsets of X.

   First we show that $A \subset B \implies A \cap (X \setminus B) = \varnothing$. Assume $A \subset B$. Just suppose A and $X \setminus B$ are *not* disjoint, that is, there is some $x \in A \cap (X \setminus B)$. Then $x \in A$ but $x \notin B$. However, since $x \in A$ and $A \subset B$, actually $x \in B$. This is a contradiction.

   Conversely, we show that $A \cap (X \setminus B) = \varnothing \implies A \subset B$. Assume that $A \cap (X \setminus B) = \varnothing$. Let $x \in A$. Then $x \in X$ since $A \subset X$. If $x \notin B$ then $x \in X \setminus B$ and so $x \in A \cap X \setminus B = \varnothing$, which is impossible; hence $x \in B$, too. $\square$