1. [Re Theorem 5, *Real Numbers*, proof of property (***).] We use induction on n. First, $b_0 - a_0 = 2 - 1 = 1/2^0$. Now let $n \in \mathbb{N}$ and assume $b_n - a_n = 1/2^n$. Now either $[a_{n+1}, b_{n+1}] = [a_n, c_n]$ or else $[a_{n+1}, b_{n+1}] = [c_n, b_n]$. Since $c_n = (a_n + b_n)/2$, we have $c_n - a_n = b_n - c_n = (b_n - a_n)/2$. Hence in either case $b_{n+1} - a_{n+1} = (b_n - a_n)/2 = (1/2^n)/2 = 1/2^{n+1}$. $\square$

2. [Complete the proof that $\mathbb{N}^*$ is infinite.] First we show that $g\colon \{1, 2 \ldots, n, n+1\} \to \mathbb{N}^*$ is injective. Let $j, k \in \{1, 2 \ldots, n, n+1\}$ with $j \neq k$. If $1 \leq j, k \leq n$, then $f(j) \neq f(k)$ since f is injective, and so

   $$g(j) = 1 + f(j) \neq 1 + f(k) = g(k).$$

   If $1 \leq j \leq n$ and $k = n + 1$, then $g(j) = 1 + f(j) > 1$ whereas $g(k) = 1$, so that $g(j) \neq g(k)$. Similarly, if $1 \leq k \leq n$ and $j = n + 1$, then $g(k) \neq g(j)$.

   Next, we show that g is surjective. Let $m \in \mathbb{N}^*$. If, on the one hand, $m = 1$, then $m = g(n+1)$. If, on the other hand, $m > 1$, then $m-1 \in \mathbb{N}^*$; in this case $m-1 = f(j)$ for some $1 \leq j \leq n$ because f is surjective, and then

   $$m = 1 + (m - 1) = 1 + f(j) = g(j).$$

   Then g is a bijection. But then the composite

   $$f^{-1} \circ g\colon \{1, 2 \ldots, n, n + 1\} \to \{1, 2 \ldots, n\}$$

   is also bijective. This contradicts Lemma 1 of *Subsets of finite sets are finite.* $\square$

3. [The union of two finite sets is finite.] Let A be finite with $m = \#(A)$. We prove that, for every natural number n, *for every finite set B with $\#(B) = n$,* the union $A \cup B$ is also finite. We use induction on n.

   Base step ($n = 0$). Let B be a finite set with $\#(B) = 0$. Then $B = \varnothing$. This means that $A \cup B = A$, so there is nothing to prove.

   Inductive step. Let $n \geq 0$ and assume that for every finite set B with $\#(B) = n$, the union $A \cup B$ is finite. Let B be a finite set with $\#(B) = n + 1$.

   If $B \subset A$, then $A \cup B = A$ and there is nothing to prove.

   Assume now that $B \not\subset A$. Then there exists some $b \in B$ with $b \notin A$. Define

   $$D = B \setminus \{b\}.$$

   Since B is finite, its subset D is also finite. Moreover, $\#(D) = n$.[1] By the inductive assumption, $A \cup D$ is finite. Now $b \notin A$ and of course $b \notin D$, so that $b \notin A \cup D$. Since

   $$A \cup B = A \cup (D \cup \{b\}) = (A \cup D) \cup \{b\},$$

   it follows from Lemma 2 of *Subsets of finite sets are finite* that $A \cup B$ is finite. $\square$

[1] Optional proof that $\#(D) = n$: Let $h\colon \{1, 2, \ldots, n, n + 1\} \approx B$. Since we could compose h with a permutation of $\{1, 2, \ldots, n, n + 1\}$ that swaps $n + 1$ with $h^{-1}(b)$, we may assume without loss of generality that $h(n + 1) = b$. Then the restriction of h to $\{1, 2, \ldots, n\}$ in the domain and $h(\{1, 2, \ldots, n\})$ in the codomain gives $\{1, 2, \ldots, n\} \approx D$.

4. [Finish proof of Lemma 4.1.36.] First, let $f \in 2^X$. By definition of φ and ψ,

$$\varphi\big(\psi(f)\big) = \varphi(\{\, x \in X : f(x) = 1 \,\}) = c_A$$

where $A = \{\, x \in X : f(x) = 1 \,\}$. Both f and c_A are functions from X to $\{0, 1\}$. Moreover, for $x \in X$,

$$f(x) = 1 \iff x \in A \iff c_A(x) = 1$$

and so also $f(x) = 0 \iff c_A(x) = 0$. Hence $f = c_A$. This means that $\varphi\big(\psi(f)\big) = f$.

Second, let $A \in \mathcal{P}(X)$. By definition of φ and ψ,

$$\psi\big(\varphi(A)\big) = \psi(c_A) = \{\, x \in X : c_A(x) = 1 \,\} = A. \qquad \Box$$

5. [Exercise 4.2.3 (1).] Let O be the set of all odd positive integers. The map $f\colon \mathbb{N} \to O$ defined by $f(n) = 2\,n + 1$ is surjective because each odd positive integer has the form $2\,n + 1$ for some natural number n; it is injective because if $f(m) = f(n)$, that is, $2\,m + 1 = 2\,n + 1$, then $m = n$.

6. [Complete the proof of Prop. 4.2.4.] By the construction of the sequence $(x_n)_{n=0}^{\infty}$,

$$i < j \implies x_i < x_j \qquad (i, j \in \mathbb{N}).$$

We use strong induction on m to show that $m \in A \implies m = x_n$ for some $n \in \mathbb{N}$.

Base step ($m = \min A$): By the construction, $\min A = x_0$.

Inductive step: Let $m \in A$ with $m > \min A$ and assume that, for each $k \in A$ with $k < m$ we have $k = x_n$ for some $n \in \mathbb{N}$.

The set $\{\, a \in A : k < m \,\}$ is finite and nonempty, so it has a greatest element b. Then $b \in A$ and $b < m$. By the inductive assumption,

$$b = x_n$$

for some $n \in \mathbb{N}$. Now $i \in \mathbb{N}$ with $i \le n$ implies $x_i \le x_n$; this means that m is the least element of $A \setminus \{x_0, x_1, \ldots, x_n\}$. But this least element is, by definition, x_{n+1}. Thus $m = x_{n+1}$. $\Box$

7. [Prove Lemma 1.] Since A is denumerable and $\mathbb{N}^* \approx \mathbb{N}$, there exists a bijection $f\colon \mathbb{N}^* \approx A$. Then the extension $F\colon \mathbb{N} \to A \cup \{b\}$ of f given by $F(0) = b$ is clearly a bijection. Hence $A \cup B = A \cup \{b\}$ is denumerable. $\Box$

8. [Prove Lemma 2.] Fix a denumerable set A. We use induction on $n = \#(B)$ to show that if B is finite with A and B disjoint, then $A \cup B$ is denumerable.

Base step. If $n = 0$, then $B = \varnothing$ and so $A \cup B = A$ is denumerable.

In the inductive step we shall need the case $n = 1$; this case is just Lemma 1.

Inductive step: Now let $n \ge 0$ and assume that, for every finite set B disjoint from A with $\#(B) = n$, the set $A \cup B$ is denumerable.

Let B be a finite set that is disjoint from A with $\#(B) = n + 1$. Choose some $b \in B$; such exists because $\#(B) > 0$. Then $B \setminus \{b\}$ is a finite set with $\#(B \setminus \{b\}) = n$, and $B \setminus \{b\}$ is disjoint from A. By the inductive assumption, $A \cup (B \setminus \{b\})$ is finite. Now

$$A \cup B = \big(A \cup (B \setminus \{b\})\big) \cup \{b\}.$$

From the case $n = 1$ proved above, it follows that $A \cup B$ is denumerable, too. $\Box$

9. [Prove Proposition 3.] Write

$$A \cup B = A \cup (B \setminus A)$$

The two sets on the right-hand side above are disjoint. Moreover, as a subset of the finite set B, the second one, $B \setminus A$, is finite. The result now follows at once from Lemma 2. □

10. [Prove Lemma 4.] There exist bijections $f\colon \mathbb{N} \approx A$ and $g\colon \mathbb{N} \approx B$. Define $h\colon \mathbb{N} \to A \cup B$ by:

$$h(n) = \begin{cases} f(n/2) & \text{if } n \text{ is even,} \\ g\big((n-1)/2\big) & \text{if } n \text{ is odd.} \end{cases}$$

Then h is a bijection. The proof of that is similar to the proof used to show that $\mathbb{Z}$ is denumerable. Details follow.

We show that h is surjective. Let $y \in A \cup B$. If $y \in A$, there exists $j \in \mathbb{N}$ with $y = f(j)$; then $y = h(2\,j)$. Similarly, if $y \in B$, there exists $k \in \mathbb{N}$ with $y = g(k)$; then $y = h(2\,k + 1)$.

We show that h is injective. Let $j, k \in \mathbb{N}$ and suppose $h(j) = h(k)$. There are three cases:

Case 1: $h(j), h(k) \in A$. Then $f(j/2) = f(k/2)$ so that $j/2 = k/2$ because f is injective, and so $j = k$.

Case 2: $h(j), h(k) \in B$. Then $j = k$ as in Case 1.

Case 3: $h(j) \in A$ and $h(k) \in B$, or vice versa. Then $h(j) = h(k) \in A \cap B$. This is impossible because A and B are disjoint. □