Math 300.2 | **Problem Set 10** | **Fall, 2007**

Due: Friday, December 7, 1:00 p.m.
Leave in Liz's LGRT 1623D mailbox

1. In the proof of Theorem 5, *The Real Numbers* notes, establish property (***) of the intervals $[a_n, b_n]$.

2. Complete the following proof (like the one begun in class) that $\mathbb{N}^*$ is infinite: Just suppose $\mathbb{N}^*$ is finite. Since $\mathbb{N}^* \neq \varnothing$, there exists some positive integer n and a bijection $f\colon \{1, 2 \ldots, n\} \approx \mathbb{N}^*$. Define a map $g\colon \{1, 2 \ldots, n, n+1\} \to \mathbb{N}^*$ by
$$g(j) = \begin{cases} 1 + f(j) & \text{if } 1 \leq j \leq n \\ 1 & \text{if } j = n+1. \end{cases}$$
Then g is a bijection because .... Then ....

3. Prove that the union $A \cup B$ of two finite sets is finite by induction on $n = \#(B)$. *Hint:* Use Lemma 2 from the note *Subsets of finite sets are finite.*

4. Complete the proof of Lemma 4.1.36 by showing that the composites $\varphi \circ \psi$ and $\psi \circ \varphi$ are actually identity maps.

5. Do Exercise 4.2.3 (1).

6. Complete the proof of Proposition 4.2.4 by using induction on m to show that $m \in A \implies m = x_n$ for some $n \in \mathbb{N}$. (*Hint:* The base step should be *not* $m = 1$, but rather $m =$ least element of A.)

Recall the lemma that if A is finite and $b \notin A$, then $A \cup \{b\}$ is finite and $\#\big(A \cup \{b\}\big) = \#(A) + 1$. That lemma was used to prove the proposition that if A and B are disjoint finite sets, then $A \cup B$ is finite and $\#(A \cup B) = \#(A) + \#(B)$. The results below concern the analogous situation where A is infinite.

7.–10. Prove the following lemmas and proposition.

Lemma 1. *Let A be a denumerable set and let $b \notin A$. Then $A \cup \{b\}$ is also denumerable.*

Lemma 2. *Let A be a denumerable set and let B be a finite set that is disjoint from A. Then $A \cup B$ is also denumerable.*

Hint: Use Lemma 1.

Proposition 3. *Let A be a denumerable set and let B be any finite set . Then $A \cup B$ is also denumerable.*

Hint: Use induction on $\#(B)$.

Lemma 4. *Let A and B be* disjoint *denumerable sets. Then $A \cup B$ is denumerable.*

Hint: Look at the proof that $\mathbb{Z}$ is denumerable.