Subsets of finite sets are finite[1]
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In order to prove that $\mathbb{N}$ is infinite, we shall need Lemma 4.1.12, restated as Lemma 1, below. The text already gives a proof of this lemma, but the proof below is perhaps a bit simpler.

Lemma 1. *For every positive integer n,*

$$\{1, 2, \ldots, n, n+1\} \not\approx \{1, 2, \ldots, n\}.$$

Proof. We use induction on n.

Base step ($n = 1$): Obviously $\{1, 2\} \not\approx \{1\}$. (Why not?)

Inductive step: Let $n \in \mathbb{N}^*$ and assume $\{1, 2, \ldots, n+1\} \not\approx \{1, 2, \ldots, n\}$. Just suppose $\{1, 2, \ldots, n+2\} \approx \{1, 2, \ldots, n+1\}$. Then there is some bijection

$$h\colon \{1, 2, \ldots, n+1, n+2\} \approx \{1, 2, \ldots, n, n+1\}.$$

The idea now is to modify h so as to make $n + 2 \mapsto n + 1$, and then restrict the modified h to a bijection $\{1, 2, \ldots, n+1\} \approx \{1, 2, \ldots, n\}$, which will give a contradiction.

To do that, first swap the elements $n + 1$ and $h(n + 2)$ of $\{1, 2, \ldots, n, n+1\}$. In other words, define the function $\sigma\colon \{1, 2, \ldots, n, n+1\} \to \{1, 2, \ldots, n, n+1\}$ by:

$$\begin{aligned} \sigma(n+1) &= h(n+2), \\ \sigma\big(h(n+2)\big) &= n+1, \\ \sigma(j) &= j \text{ if } j \neq h(n+2) \text{ and } j \neq n+1. \end{aligned}$$

Then σ is a permutation of $\{1, 2, \ldots, n, n+1\}$, that is, a bijection from the set $\{1, 2, \ldots, n, n+1\}$ to itself.

Now define

$$h' = \sigma \circ h,$$

so that

$$h'\colon \{1, 2, \ldots, n+1, n+2\} \to \{1, 2, \ldots, n, n+1\}$$

Then h' is also bijective (why?), and

$$h'(n+2) = n+1.$$

Hence we may restrict the domain of h' to $\{1, 2, \ldots, n+1\}$ and the codomain of h' to $\{1, 2, \ldots, n\}$ to get a bijection $f\colon \{1, 2, \ldots, n+1\} \approx \{1, 2, \ldots, n\}$. The existence of such an f contradicts the induction hypothesis. □

In our work with finite sets, we shall concentrate only upon whether sets are finite and not upon how many elements they have when they are finite. Then we shall not need the more general result of Corollary 4.1.3 that $\{1, 2, \ldots, m\} \not\approx \{1, 2, \ldots, n\}$ whenever $m \neq n$. Accordingly, we give below a slightly simpler proof than the one in the text that a subset of a finite set is finite.

Lemma 2. *If A is a finite set and if $b \notin A$, then $A \cup \{b\}$ is finite.*

Proof. Assume A is finite and $b \notin A$. If $A = \emptyset$, then $A \cup \{b\} = \{b\} \approx \{1\}$, so that $A \cup \{b\}$ is finite.

Now assume $A \neq \emptyset$. Then there is some positive integer n and some bijection $f\colon A \to \{1, 2, \ldots, n\}$. Extend f to a function $F\colon A \cup \{b\} \to \{1, 2, \ldots, n, n+1\}$ by defining

$$F(x) = \begin{cases} f(x) & \text{if } x \in A, \\ n+1 & \text{if } x = b. \end{cases}$$

Then F is a bijection (evidently F is surjective, and F is injective because f is injective, $b \notin A$, and $n + 1 \notin \{1, 2, \ldots, n\}$). Hence $A \cup \{b\}$ is finite. □

Proposition 3. *Every subset B of a finite set A is finite.*

Proof. Since the only subset of the empty set is empty, the result is trivial when $A = \emptyset$. If A is a nonempty finite set and B is a subset of A, there is some positive integer n and some bijection $f\colon A \to \{1, 2, \ldots, n\}$; in this case, $f(B)$ is a subset of $\{1, 2, \ldots, n\}$, and B will be finite if and only if $f(B)$ is finite. Hence it suffices to prove that, for every positive integer n, every subset of $\{1, 2, \ldots, n\}$ is finite.

We use induction on n.

Base step ($n = 1$): The only subsets of $\{1\}$ are $\emptyset$ and $\{1\}$ itself, and both are finite.

Inductive step: Let n be a positive integer and assume that every subset of $\{1, 2, \ldots, n\}$ is finite. Let $B \subset \{1, 2, \ldots, n, n+1\}$. It remains to deduce that B is finite.

If $n + 1 \notin B$, then actually $B \subset \{1, 2, \ldots, n\}$, so that B is finite by the inductive assumption.

Assume now that $n + 1 \in B$. Define $D = B \setminus \{n + 1\}$. Then $D \subset \{1, 2, \ldots, n\}$. By the inductive assumption, D is finite. Since $B = D \cup \{n + 1\}$ and $n + 1 \notin D$, from the Lemma it follows that B is finite, too. □