The real numbers[1]
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What are the real numbers? As explained in the introduction to Chapter 3, there are two approaches to answering this question. One approach would be to construct the set $\mathbb{R}$ of real numbers starting with the set $\mathbb{N}$ of natural numbers. Along the way, we would need to construct the set $\mathbb{Z}$ of integers and the set $\mathbb{Q}$ of rational numbers.

The other approach—the one followed here—is not to say what the real numbers are, but only to describe how they behave. In other words, our approach is to state, in the form of axioms, the fundamental properties of the set of real numbers.

Our axiomatic approach does not, of course, establish that any set having these properties actually exists! To establish that one needs to take a constructive approach. But the axiomatic approach does uniquely characterize the set of real numbers in the sense that any two sets having these properties are "essentially" the same (in technical terms, they are "order isomorphic" as ordered fields). This essential uniqueness is established in Section 3.4. However, in this course we shall not look into the matter of uniqueness of $\mathbb{R}$.

The axioms for $\mathbb{R}$ can be summarized as follows:

> $\mathbb{R}$ ***is an ordered field that includes*** $\mathbb{N}$***, has the Archimedean Ordering Property, and has the Nested Interval Property.***

To say that $\mathbb{R}$ is a **field** means that there are operations of addition and multiplication in $\mathbb{R}$ with "the usual" algebraic properties. These algebraic properties are listed in Axioms 3.1.4. In particular, each $x \in \mathbb{R}$ has a negative $-x$ for which $x + (-x) = 0$, and each *nonzero* $y \in \mathbb{R}$ has a multiplicative inverse y^{-1} for which $yy^{-1} = 1$. Then subtraction and division of real numbers may be defined by $x - y = x + (-y)$ and, for $y \neq 0$, $x/y = xy^{-1}$.

Beyond those basic properties we shall assume without proof all the familiar algebraic properties listed in Proposition 3.1.12.

To say that the field $\mathbb{R}$ is ordered means that there is a relation $<$ in $\mathbb{R}$ having "the usual" algebraic properties such as those listed in Proposition 3.2.4. Among these properties are the following:

- For each $x \in \mathbb{R}$, exactly one of the relations $0 < x$, $x = 0$, and $x < 0$ holds.
- For all $x, y \in \mathbb{R}$, $x < y \iff 0 < y - x$.
- For all $x, y \in \mathbb{R}$, if $0 < x$ and $0 < y$, then $0 < x + y$ and $0 < xy$.

From just these three properties all the other familiar properties of order for real numbers (as listed in Proposition 3.2.4) can be deduced. Here's an example: For all $x, y, z \in \mathbb{R}$,

$$x < y \implies x + z < y + z.$$

In fact, assume $x < y$. This means $0 < y - x$. But $y - x = (y + z) - (x + z)$. Thus $0 < (y + z) - (x + z)$. This means $x + z < y + z$.

Exercise 1. Using just the properties of ordering listed above together with the usual algebraic properties of addition and multiplication, deduce:

(a) If $0 < x$, then $-x < 0$.

(b) If $x < y$, then $-y < -x$.

(c) $0 < 1$

When we say here that $\mathbb{R}$ "includes" $\mathbb{N}$, we do not mean merely that $\mathbb{N} \subset \mathbb{R}$. Rather, we mean also that:

- if $m, n \in \mathbb{N}$, then their sum $m + n$ and product $m \cdot n$ as elements of $\mathbb{R}$ are the same as their sum and product, respectively, as natural numbers [as defined in Example 1.2.10 (2) and Exercise 1.2.11 (3)];
- the number $0 \in \mathbb{N}$ is the identity element for addition in the field $\mathbb{R}$ and the number $1 \in \mathbb{N}$ is the identity element for multiplication in the field $\mathbb{R}$; and
- if $m, n \in \mathbb{N}$, then the relation $m < n$ holds for m and n as elements of $\mathbb{R}$ if and only if it holds for them as elements of $\mathbb{N}$ [as defined in Exercise 1.2.11 (2)].

With $\mathbb{N}$ as a subset of $\mathbb{R}$, the set $\mathbb{Z}$ of all integers and the set $\mathbb{Q}$ of all rational numbers may be defined as:

$$\mathbb{Z} = \mathbb{N} \cup \{\, -n : n \in \mathbb{N} \,\},$$
$$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{N}, n \in \mathbb{N}, n \neq 0 \right\}$$

The **Archimedean Ordering Property** of the ordered field $\mathbb{R}$ is the following:

> For every $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ and for each $c \in \mathbb{R}$, there exists a positive integer n such that $n\varepsilon > c$.

This property may be expressed by saying that if you take a real number c, no matter how large, and a positive real number ε, no matter how small, then if you add ε to itself enough times, the sum will be greater than c.

For more about the Archimedean Ordering Property, see Section 3.2.

Proposition 2. *Let $c \in \mathbb{R}$. Then there exists a unique $n \in \mathbb{Z}$ for which $n \leq c < n + 1$.*

Proof. Existence. Case (i): $c \geq 0$. By the Archimedean Ordering Property, there exists some $k \in \mathbb{N}$ such that $k \cdot 1 > c$. By the Well-Ordering Principle for $\mathbb{N}$, there is a least such k; call it k_1. Let $n = k_1 - 1$. Then $n \leq c < n + 1$ (why?).

Case (ii): $c < 0$. Then $-c > 0$ and so, by what was just proved, there exists an integer m with $m \leq -c < m + 1$. Then …(finish the existence proof in this case.)

Uniqueness. Exercise. □

Theorem 3 (Order Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *If $a, b \in \mathbb{R}$ with $a < b$, there exists some $q \in \mathbb{Q}$ for which $a < q < b$.*

Proof. See the proof of Theorem 3.2.27. In that proof, just change F to $\mathbb{R}$, $Q(F)$ to $\mathbb{Q}$, and $N(F)$ to $\mathbb{N}$. □

For real numbers a and b, as usual the closed interval $[a, b]$ is the set $\{\, x \in \mathbb{R} : a \leq x \leq b \,\}$.

The **Nested Interval Property** of $\mathbb{R}$ is the following:

> If $([a_n, b_n])_{n=0,1,2\ldots}$ is a sequence of closed intervals in $\mathbb{R}$ that is decreasing in the sense that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for each $n = 0, 1, 2, \ldots$, then $\bigcap_{n=0}^{\infty} [a_n, b_n] \neq \emptyset$.

For more about the Nested Interval Property, see Section 3.3.

We already know there is no *rational number* c for which $c^2 = 2$. One of the consequences of the Nested Interval Property is that there is some *real* number c for which $c^2 = 2$ or, as we shall say, that $\sqrt{2}$ exists.

With the theory of calculus at our disposal this would be easy to prove: see Exercise 4

Exercise 4. Prove that $\sqrt{2}$ exists by applying the Intermediate Value Theorem to the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.

For the method in Exercise 4, you need to know that the function f is continuous (and so you need to know a precise definition for "continuous"); you also need to know have the Intermediate Value Theorem at your disposal. So we shall not use such a proof.

That $\sqrt{2}$ exists is proved by an indirect method in Section 3.3, namely, by first deducing from the Archimedean Ordering Property and the Nested Interval Property that each nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

In this note we give a different proof that directly invokes the Archimedean Ordering Property and the Nested Interval Property. In following the proof it may help if you draw a diagram with a line representing $\mathbb{R}$ and mark on it the points and intervals constructed. Be sure to supply any missing justifications for steps.

Theorem 5. *There exists a real number c for which $c^2 = 2$.*

Proof. Define a sequence $([a_n, b_n])_{n=0,1,2\ldots}$ of closed intervals in $\mathbb{R}$ recursively as follows. First, let

$$[a_0, b_0] = [1, 2]$$

and let

$$c_0 = \text{the midpoint of } [a_0, b_0].$$

Next, let

$$[a_1, b_1] = \begin{cases} [a_0, c_0] & \text{if } 2 < c_0^2, \\ [c_0, b_0] & \text{if } c_0^2 < 2. \end{cases}$$

(We cannot have $c_0^2 = 2$ because no rational number has square 2.) In general, once $[a_n, b_n]$ has been constructed, let

$$c_n = \text{the midpoint of } [a_n, b_n]$$

and then let

$$[a_{n+1}, b_{n+1}] = \begin{cases} [a_n, c_n] & \text{if } 2 < c_n^2, \\ [c_n, b_n] & \text{if } c_n^2 < 2. \end{cases}$$

(Since no rational number has square 2, in fact the case $c_n^2 = 2$ never arises.)

By the construction, $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n = 0, 1, 2, \ldots$. Then

$$a_n < b_n \leq 2, \tag{*}$$

$$2 \in [a_n^2, b_n^2], \tag{**}$$

$$b_n - a_n = \frac{1}{2^n} \tag{***}$$

for all $n = 0, 1, 2, \ldots$. (Why?)

By the Nested Interval Property of $\mathbb{R}$, there exists some c with

$$c \in \bigcap_{n=0}^{\infty} [a_n, b_n].$$

We are going to show that $c^2 = 2$.

Just suppose $c^2 \neq 2$. Define

$$\varepsilon = |c^2 - 2|,$$

so that $\varepsilon > 0$. By the Archimedean Ordering Property of $\mathbb{R}$, there exits a positive integer n for which

$$\frac{1}{2^{n-2}} < \varepsilon.$$

(Why?)

Fix such an n. Since both c^2 and 2 are in $[a_n^2, b_n^2]$, then

$$\begin{aligned}|c^2 - 2| &\le b_n^2 - a_n^2 = (b_n + a_n)(b_n - a_n)\\ &< (2+2)\frac{1}{2^n} = \frac{1}{2^{n-2}}\\ &< \varepsilon\end{aligned}$$

This is impossible because $|c^2 - 2| = \varepsilon$. $\square$

Exercise 6. Use the method of the preceding proof to establish that $\sqrt{3}$ exists.

Exercise 7. Prove uniqueness of the number c whose existence is guaranteed by Theorem 5.

The method of proof of Theorem 5 does not merely establish existence of a real number c whose square is 2. It also provides a method for approximating that number as closely as we wish. Indeed, look again at the construction of the intervals $[a_n, b_n]$. The number c we want belongs to each of these intervals (and is different from the endpoints of each). Initially, all we know about c is that $c \in (a_0, b_0) = (1, 2)$, that is,

$$1 < c < 2.$$

If we use the midpoint $c_0 = 1.5$ as an approximation to c, the the error in that approximation—the size $|c - c_0|$ of the difference between c and the approximation c_0—satisfies

$$|c - c_0| < 0.5$$

(half the length of $[a_0, b_0] = [1, 2]$). Next, since $c_0^2 = 1.5^2 = 2.25 > 2$, then $[a_1, b_1] = [a_0, c_0] = [1, 1.5]$. So now we know $c \in (1, 1.5)$, that is,

$$1 < c < 1.5.$$

If we use the midpoint $c_1 = 1.25$ as a new approximation to c, then the error $|c - c_1|$ in this approximation satisfies

$$|c - c_1| < 0.25$$

(half the length of $[a_1, b_1] = [1, 1.5]$). Next, since $c_1^2 = 1.25^2 = 1.5625 < 2$, then $[a_2, b_2] = [c_1, b_1] = 1.25, 1.5]$. So now we know that $c \in (1.25, 1.5)$, that is,

$$1.25 < c < 1.5.$$

If we use the midpoint $c_2 = 1.125$ as a new approximation to c, then the error $|c - c_2|$ in this approximation satisfies

$$|c - c_2| < 0.125.$$

With each successive bisection of the interval, the number c is trapped inside an interval of half the length of the interval used in the previous step, and the error in the approximating midpoint is halved. This method is known as the "Bisection Method".

Exercise 8. Continue the Bisection Method begun above so as to approximate $\sqrt{2}$ with an error less than 0.01.

The method of proof used for Theorem 5—constructing a nested sequence of closed intervals—is applicable for the next theorem, too.

Recall the relevant definitions (which we originally gave in $\mathbb{N}$). Let $A \subset \mathbb{R}$. An **upper bound** of A in $\mathbb{R}$ is a number $b \in \mathbb{R}$ such that $a \le b$ for every $a \in A$. The set A is **bounded above** in $\mathbb{R}$ when there exists some upper bound of A in $\mathbb{R}$.

About $\mathbb{N}$ you proved that each nonempty subset A of $\mathbb{N}$ that is bounded above in $\mathbb{N}$ has a greatest element. Such a greatest element g of A is an upper bound of A in $\mathbb{N}$ that is no greater than any other upper bound of A in $\mathbb{N}$; in other words, g is the least element of the set of all upper bounds of A in $\mathbb{N}$. So this greatest element g of A is *least upper bound* of A in $\mathbb{N}$.

A subset of $\mathbb{R}$ that is bounded above in $\mathbb{R}$ need not have a greatest element. For example, the open interval $(0, 1)$ in $\mathbb{R}$ has no greatest element (why?). But it does have a least upper bound in $\mathbb{R}$, namely, the real number 1. The following theorem generalizes this example.

Theorem 9 (Order Completeness of $\mathbb{R}$)**.** *Each nonempty subset of* $\mathbb{R}$ *that is bounded above has a least upper bound.*

Proof. See the proof of Theorem 3.3.13. In that proof, just change F to $\mathbb{R}$ and change the term "supremum" to "least upper bound". □