Proof of the Gap Lemma[1]
October 22, 2007, 11:26am

This note provides a proof of the Gap Lemma based ultimately upon the Peano Postulates. It collects all the requisite preliminary results about $\mathbb{N}$.

The Gap Lemma. *For each natural number m, there exists* no *natural number n for which $m < n < n + 1$.*

Lemma 1. *For each natural number $m \neq 0$, there exists a natural number k such that $m = k + 1$.*

Proof. This follows from the consequence $\sigma(\mathbb{N}) = \mathbb{N}^*$ of the Peano Postulates. □

Recall the notation $n + 1 = \sigma(n)$ for a natural number n.

Lemma 2. *If $k_1, k_2 \in \mathbb{N}$ and $k_1 + 1 = k_2 + 1$, then $k_1 = k_2$.*

Proof. This just restates the Peano Postulate that $\sigma\colon \mathbb{N} \to \mathbb{N}$ is injective. □

According to Lemmas 1 and 2, for each natural number $m \neq 0$, there is *exactly one* natural number k such that $m = k + 1$. This justifies the following definition.

Definition 3. Let m be a natural number with $m \neq 0$. By $m - 1$ we denote the unique natural number k for which $k + 1 = m$. In other words, the natural number $m - 1$ is uniquely defined by:

$$(m - 1) + 1 = m$$

The recursive definition of addition of natural numbers is as follows [see Example 1.2.10 (2)].

Definition 4. For each $m \in \mathbb{N}$:

$$\begin{cases} m + 0 = m \\ m + (n + 1) = (m + n) + 1 \qquad (n \in \mathbb{N}) \end{cases}$$

Proposition 5. *Addition in $\mathbb{N}$ is associative and commutative.*

Proof. See Exercises 1.2.11 (a)–(c). □

The "strict" order relation $<$ in $\mathbb{N}$ and the associated "weak" order relation $\leq$ in $\mathbb{N}$ are defined as follows [see Exercise 1.2.11 (2)]:

Definition 6. Let $m, n \in \mathbb{N}$. Then $m < n$ is defined to mean there exists some $d \in \mathbb{N}^*$ for which $m + d = n$. And $m \leq n$ is defined to mean $m < n$ or $m = n$.

Remark. Let $m, n \in N$. Then $m \leq n$ if and only if there exists some $d \in \mathbb{N}$ for which $m + d = n$.

Lemma 7. *The relation $<$ in $\mathbb{N}$ has the properties:*

1. (transitivity) *For all $m, n, k \in \mathbb{N}$, if $m < n$ and $n < k$, then $m < k$.*
2. (irreflexivity) *For all $m \in \mathbb{N}$, we have $m \not< m$.*
3. (asymmetry) *for all $m, n \in \mathbb{N}$, if $m < n$, then $n \not< m$.*

Proof. 1. Let $m, n, k \in \mathbb{N}$ with $m < n$ and $n < k$. There exists $s, t \in \mathbb{N}^*$ for which $m + s = n$ and $n + t = k$, respectively. Then $m + (s + t) = k$. Of course $s + t \in \mathbb{N}$. Moreover, $s + t \neq 0$. In fact, $s + t = s + \big((t-1) + 1\big) = \big(s + (t-1)\big) + 1 \neq 0$ because it is the successor of a natural number. [Here we used the Peano Postulate that $0 \notin \sigma(\mathbb{N})$.]

2. This is an exercise.

3. Let $m, n \in \mathbb{N}$ with $m < n$. Just suppose $n < m$. By transitivity, $m < m$. But this contradicts irreflexivity. □

Lemma 8. *The weak relation $\leq$ in $\mathbb{N}$ is a partial ordering of $\mathbb{N}$.*

Proof. First, $\leq$ is reflexive: $m \in \mathbb{N}$ implies $m \leq m$ because, in fact, $m = m$.

Second, $\leq$ is transitive: this is an easy consequence of transitivity of $<$.

Third, $\leq$ is antisymmetric: Let $m, n \in \mathbb{N}$ with $m \leq n$ and $n \leq m$. Just suppose, though, that $m \neq n$. Since $m \leq n$, then $m < n$; similarly, $n < m$. This contradicts irreflexivity of $<$ (Lemma). □

The next result says that addition of natural numbers "preserves order". [The first part is Exercise 1.2.11 (2).]

Proposition 9. *Let $m, n \in \mathbb{N}$. Then:*

$$\begin{aligned} m \leq n &\implies m + k \leq n + k \textit{ for all } k \in \mathbb{N}, \\ m < n &\implies m + k < n + k \textit{ for all } k \in \mathbb{N}. \end{aligned}$$

Proof. Assume $m < n$. By definition, there exists $d \in \mathbb{N}^*$ with $m + d = n$. Then for each $k \in \mathbb{N}$, we have $(m + k) + d = (m + d) + k = n + k$ and so $m + k < n + k$ by definition of $<$ again.

The proof of the result about $m \leq n$ is left as an exercise. □

Proposition 10. *The relation $\leq$ is a total ordering of $\mathbb{N}$.*

Proof. According to Lemma 8, already $\leq$ partially orders $\mathbb{N}$. It remains only to prove that $\leq$ has the comparability property: $m, n \in \mathbb{N} \Longrightarrow m \leq n$ or $n \leq m$. We use induction on n to prove that, for each $n \in \mathbb{N}$:

$$(\forall m \in \mathbb{N})(m \leq n \text{ or } n \leq m) \tag{*}$$

Base step $(n = 0)$: If $m \in \mathbb{N}$, then $m = 0 + m$ so that $0 \leq m$.

Inductive step: Let $n \in \mathbb{N}$ an assume (*). Let $m \in \mathbb{N}$. We wish to deduce that $m \leq n + 1$ or $n + 1 \leq m$.

If $m = 0$, then already $m = 0 \le n + 1$.

Now suppose $m \ne 0$. There exists $k \in \mathbb{N}$ with $m = k + 1$. By the inductive assumption, $k \le n$ or $n \le k$. If, on the one hand, $k \le n$, then $m = k + 1 \le n + 1$ and so $m \le n + 1$. If, on the other hand, $n < k$, then $n + 1 < k + 1 = m$, so that $n + 1 < m$, and a fortiori $n + 1 \le m$. □

Lemma 11. *For each natural number n, we have $n \nless 0$.*

Proof. Just suppose there is some natural number n for which $n < 0$. By definition of $<$, this means there is some $k \in \mathbb{N}*$ for which $n + k = 0$. □

The final lemma here is the crux of the Gap Lemma: *there is no natural number strictly between* 0 *and* 1.

Lemma 12. *There exists* no *natural number n for which $0 < n < 1$.*

Proof. What is to be proved is $\neg(\forall n \in \mathbb{N})(0 < n < 1)$. Now $0 < n < 1$ means $0 < n \,\&\, n < 1$. Thus what is to be proved is, equivalently:

$$(\forall n \in \mathbb{N})(n \le 0 \text{ or } 1 \le n)$$

We use induction on n.

Base step $(n = 0)$: Since $0 \le 0$, certainly $0 \le 0$ or $1 \le 0$.

Inductive step: Now let $n \in \mathbb{N}$ and assume $n \le 0$ or $1 \le n$. There are two cases.

Case (i): $n \le 0$. In this case, by the preceding lemma $n = 0$. Then $1 = n + 1$, which means $1 \le n + 1$.

Case (ii): $1 \le n$. In this case, also $1 \le n + 1$.

Thus in either case actually $1 \le n + 1$. □

Proof of Gap Lemma. We shall prove, equivalently, that, for each $m \in \mathbb{N}$:

$$(\forall n \in \mathbb{N})(n \le m \text{ or } m + 1 \le n).$$

(Compare the logical analysis at the start of the proof of the preceding lemma.) We use induction on m.

Base step $(m = 0)$: This is the assertion of Lemma 12

Inductive step: Now let $m \in \mathbb{N}$ and assume

$$(\forall n \in \mathbb{N})(n \le m \text{ or } m + 1 \le n). \tag{**}$$

We want to deduce $(\forall n \in \mathbb{N})(n \le m + 1 \text{ or } m + 1 \le n)$. Let $n \in \mathbb{N}$. By the inductive assumption (**), $n \le m$ or $m + 1 \le n$. There are two cases. If, on the one hand, $n \le m$, then certainly $n \le m + 1$. If, on the other hand, $m + 1 \le n$, then also $m + 1 \le n + 1$. Thus in either case $n \le m + 1$ or $m + 1 \le n$. This completes the proof of the inductive step. □