Math 300.2 **Review questions for Exam 3** **December 11, 2007**

The exam covers the material from Problem Sets 9–10 and involves the following content.

- Fermat's Little Theorem: Section 2.4, items 2.4.29–2.4.18 and 2.4.29–2.4.32
- Congruence classes and $\mathbb{Z}_m$: Section 2.5
- Equivalence relations and partitions: Sections C.1–C.2 through C.2.6.
- The ordered field $\mathbb{R}$ and its Archimedean Ordering Property, Nested Interval Property, and their consequences: note "The Real Numbers" (see also material from sections 3.1 and 3.2 covered in lecture)
- Finite and infinite sets: Chapter 4 through section 4.2 and as much of section 4.3 as covered through Tuesday, Dec. 11.

Define:

- A congruence class modulo m (where m is an integer > 1).
- The set $\mathbb{Z}_m$.
- The sum $A + B$ and the product $A \cdot B$ of two elements A and B of $\mathbb{Z}_m$.
- An equivalence relation $\sim$ on a set X.
- A partition of a set X.
- For an equivalence relation $\sim$ on a set X, the quotient set $X/\sim$.
- For sets A and B, $A \approx B$.
- A set is finite; a set is infinite; a set is denumerable; a set is countable; a set is uncountable.
- For a finite set A, $\#(A)$.
- The product $\prod_{i \in I} A_i$ of a family $(A_i)_{i \in I}$ of sets.
- The set A^I for sets A and I; the set 2^I for a set I.
- For sets A and B: $A \preccurlyeq B$, $A \prec B$.
- For a set A, $\text{card}(A) = \aleph_0$.
- For sets A and B: $\text{card}(A) = \text{card}(B)$; $\text{card}(A) \leq \text{card}(B)$; $\text{card}(A) < \text{card}(B)$

State:

- Fermat's Little Theorem
- The Archimedean Ordering Property of $\mathbb{R}$.
- The Nested Interval Property of $\mathbb{R}$.
- Cantor's Theorem.

Calculate, list, or give explicitly:

- The set of elements of the congruence class $[3] \in \mathbb{Z}_7$.
- The addition table for $\mathbb{Z}_4$; the multiplication table for $\mathbb{Z}_3$.
- If E is the set of all even integers and O is the set of all odd integers, the equivalence relation $\sim$ on $\mathbb{Z}$ for which the quotient set $\mathbb{Z}/\sim$ is $\{E, O\}$.
- The sets $\bigcup_{i=2}^{\infty} [1/i, 1-1/i]$, $\bigcup_{i=2}^{\infty} (1/i, 1-1/i)$, $\bigcap_{i=2}^{\infty} [1/i, 1-1/i]$, and $\bigcap_{i=2}^{\infty} (1/i, 1-1/i)$. (*Corrected.*)
- Several different sets that match $\mathbb{N}$.
- Several different sets that match $\mathbb{R}$.

Complete each statement:

- The union of two finite sets is ....
- If A has some denumerable subset, then ....
- The union of two denumerable sets is ....
- If A has some infinite subset, then ....
- The union of a finite set and a denumerable set is ....

Prove:

1. For $A, B \in \mathbb{Z}_m$, $A \cap B \neq \varnothing \iff A = B$.
2. Addition in $\mathbb{Z}_m$ is commutative; multiplication in $\mathbb{Z}_m$ is distributive over addition, that is, $A \cdot (B + C) = A \cdot B + A \cdot C$ for all $A, B, C \in \mathbb{Z}_m$.
3. The integer $m > 1$ is prime if and only if for each $A \in \mathbb{Z}_m$ with $A \neq [0]$, there exists $B \in \mathbb{Z}_m$ with $A \cdot B = [1]$.
4. If $\sim$ is an equivalence relation on a set X, then the set of all its equivalence classes is a partition of X.
5. For each real $\varepsilon > 0$, there exists a positive integer n with $1/n < \varepsilon$.
6. If $x \in \mathbb{R}$ with $x \geq 0$, then there exists a unique integer n such that $n \leq x < n + 1$.
7. The set $\mathbb{Q}$ is order dense in $\mathbb{R}$.
8. $\{1, 2\} \not\approx \{1\}$.
9. The union of two disjoint finite sets is finite.
10. Any subset of a finite set is finite.
11. The union of finitely many finite sets is finite.
12. The product of two finite sets is finite.
13. If A is finite and if there exists a surjection $f\colon A \to B$, then B is finite.

14. For each positive integer n, $\{1, 2\ \ldots, n+1\} \not\approx \{1, 2, \ldots, n\}$.

15. The set $\mathbb{N}$ is infinite.

16. An infinite subset of a countable set is denumerable.

17. Each infinite set has a denumerable subset.

18. A set A is infinite if and only if $A \approx C$ for some $C \subset A$ with $C \neq A$.

19. If A is countable and if there exists a surjection $f\colon A \to B$, then B is countable.

20. The set $\mathbb{N} \times \mathbb{N}$ is denumerable.

21. The set $\mathbb{Z}$ is denumerable.

22. The set $\mathbb{Q}$ is denumerable.

23. The product of two denumerable sets is denumerable.

24. The union of two denumerable sets is denumerable.

25. The union of countably many denumerable sets is denumerable (except in the trivial case that it is empty).

26. Any two closed intervals $[a, b]$ and $[c, d]$ match one another.

27. For any set X, $X \prec \mathcal{P}(X)$.

28. $(0, 1) \approx \mathbb{R}$.

29. The closed interval $[0, 1]$ in $\mathbb{R}$ is uncountable.

30. The set $\mathbb{R}$ is uncountable.

31. The set of all irrational numbers is uncountable.

32. The plane $\mathbb{R} \times \mathbb{R}$ is uncountable.

Give an example of the following or else explain why none exists:

- A relation in $\mathbb{Z}$ that is *not* an equivalence relation.
- A set X and a subset $\mathcal{A}$ of $\mathcal{P}(X)$ with $X = \bigcup_{A \in \mathcal{A}} A = X$ and $\varnothing \notin \mathcal{A}$ but $\mathcal{A}$ is *not* a partition of X.
- An infinite set that is not denumerable.
- A denumerable set that is not infinite.
- A denumerable set A and a denumerable subset B of A with $B \neq A$ but $A \setminus B$ is still denumerable.
- An uncountable set.
- An uncountable set whose cardinality is not the same as the cardinality of $\mathbb{R}$.

Which of the following relations are equivalence relations?

1. In the set $\mathbb{R}$, let $x \sim y$ mean $y - x \in \mathbb{Z}$. Also, if $\sim$ is an equivalence relation, what are $[1]$ and $[\sqrt{2}]$?
2. In the set $\mathbb{R}$, let $x \sim y$ mean $x^2 = y^2$. Also, if $\sim$ is an equivalence relation, what are $[1]$ and $[\sqrt{2}]$?
3. In the set $\mathcal{P}(\mathbb{Z})$, let $A \sim B$ mean $A \cap B \neq \varnothing$. Also, if $\sim$ is an equivalence relation, what is $[\{1\}]$?
4. In $\mathbb{Z} \times \mathbb{Z}$, let $(m, n) \sim (i, j)$ mean $m\,j = n\,i$. Also, if $\sim$ is an equivalence relation, what are $[(1, 2)]$ and $[(2, 3)]$?