Math 300.2 **Exam 3 Answers** **December 13, 2007**

1. (a) [6%] For every $c \in \mathbb{R}$ and for every $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, there exists some nonnegative integer n such that $n\,\varepsilon > c$.

   (b) [7%] In the Archimedean Ordering Property, take $\varepsilon = 1$ to obtain a nonnegative integer n with $n = n \cdot 1 > c$. Now let $k = n + 1$ to ensure that k is actually a positive integer.

   (c) [7%] According to (b), there exists some positive integer k such that $k > c$. By the Well-Ordering Principle, there exists a least such k; call it k_1. Define

   $$n = k_1 - 1.$$

   Then n is a nonnegative integer (because k_1 is a positive integer). We consider two cases.

   Case 1: $(n = 0)$. In this case, $1 = k_1 > c$ and so already $0 \le c < 0+1 = 1$.

   Case 2: $(n > 0)$. Suppose now $n > 0$, that is, n is a positive integer. Since $n < k_1$ and k_1 is the *least* positive integer k for which $k > c$, then $n \not> c$, so that $n \le c$. Thus $n \le c < k_1 = n + 1$. $\square$

2. (a) [10%] Take, for example, $m = 4$, $A = [2]$. Then $A \neq [0]$. Moreover,

   $$[2][0] = [0], \quad [2][1] = [2], \quad [2][3] = [6] = [2], \quad [2][4] = [8] = [0].$$

   Thus $A \cdot B \neq [1]$ for all $B \in \mathbb{Z}_4$.

   (b) [10%] In view of Cantor's Theorem, the power set $X = \mathcal{P}(\mathbb{R})$ has $\operatorname{card}(X) > \operatorname{card}(\mathbb{R})$. Or take $X = 2^{\mathbb{R}}$, which has the same cardinality as $\mathcal{P}(\mathbb{R})$.

   (Note: $\mathbb{R} \times \mathbb{R}$ would *not* be correct: it has the same cardinality as does $\mathbb{R}$!)

3. (a) [5%] The set A is said to be finite when $A = \varnothing$ or else there is some positive integer n such that $\{1, 2, \ldots, n\} \approx A$ (in other words, there exists some bijection $\{1, 2, \ldots, n\} \to A$).

   (b) [15%] Assume A is finite and $b \notin A$. If $A = \varnothing$, then $A \cup \{b\} = \{b\} \approx \{1\}$, which is finite. [2%]

   Suppose now that $A \neq \varnothing$. Then there exists some positive integer n and some bijection $f\colon \{1, 2, \ldots, n\} \to A$. [3%] Define

   $$g\colon \{1, 2, \ldots, n, n+1\} \to A \cup \{b\}$$

   to be the extension of f for which $g(n+1) = b$; in other words,

   $$g(k) = \begin{cases} f(k) & \text{if } 1 \le k \le n, \\ b & \text{if } k = n+1. \end{cases} \qquad [8\%]$$

   We shall show that g is a bijection.

   The map g is surjective because $g(\{1, 2, \ldots, n\}) = f(\{1, 2, \ldots, n\}) = A$ and $g(n+1) = b$. [1%]

   The map g is injective because f is injective and because if $1 \le k \le n$, then $g(k) = f(k) \in A$ whereas $g(n+1) = b \notin A$. [1%] $\square$

4. (a) [10%] The idea is to map $0, 2, 4, 6, \ldots$ to $0, 1, 3, 5, \ldots$, respectively and to map $1, 3, 5, \ldots$ to $-1, -2, -3, \ldots$, respectively. Define $f\colon \mathbb{N} \to \mathbb{Z}$ by:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ -(n+1)/2 & \text{if } n \text{ is odd.} \end{cases} \quad [8\%]$$

(This does really define a map into $\mathbb{Z}$. For if n is even, then $n/2$ is an integer—a nonnegative integer, in fact; and if n is odd, then $n+1$ is even so that $-(n+1)/2$ is an integer—a negative integer, in fact.)

We show that f is injective. Let $m, n \in \mathbb{N}$ with $m \neq n$. Clearly $f(m) \neq f(n)$ in case m and n are both even or both odd. Suppose now that m is even but n is odd. Then $f(m) \geq 0$ whereas $f(n) < 0$, and so $f(m) \neq f(n)$. Similarly if m is odd but n is even. [1%]

We show that f is surjective. Let $m \in \mathbb{Z}$. If, on the one hand, $m \geq 0$, then $2m \in \mathbb{N}$ with $f(2m) = m$ since $2m$ is even. If, on the other hand, $m < 0$, then $-(2m+1) \in \mathbb{N}$ with $f\bigl(-(2m+1)\bigr) = m$ since $-(2m+1)$ odd. [1%]

Instead of defining f, you could have defined its inverse $g\colon \mathbb{Z} \to \mathbb{N}$ by

$$g(m) = \begin{cases} 2m & \text{if } m \geq 0, \\ -(2m+1) & \text{if } m < 0, \end{cases}$$

and verified that g is bijective.

(b) [10%]

- Since $\mathbb{Z} \approx \mathbb{N}$, then $\mathbb{Z}^* \approx \mathbb{N}^* \approx \mathbb{N}$, whence $\mathbb{Z}^*$ is also denumerable. [1%]
- The product $\mathbb{Z} \times \mathbb{Z}^*$ of the two denumerable sets $\mathbb{Z}$ and $\mathbb{Z}^*$ is also denumerable. [2%]
- The map $f\colon \mathbb{Z} \times \mathbb{Z}^* \to \mathbb{Q}$ given by $f(m, n) = m/n$ is surjective (in view of what is meant by a rational number. [3%]
  [*Note:* The map f is definitely *not* bijective! E.g., $f(3, 6) = 1/2 = f(1, 2)$.]
- A proposition says that the range $\mathbb{Q}$ of the map f is *countable*. [2%]
- But $\mathbb{Q}$ is infinite (since, e.g., it contains the infinite set $\mathbb{N}$). [2%]

Since $\mathbb{Q}$ is countable but infinite, it is denumerable. $\square$

5. (a) [5%]
   (i) (reflexive) for each $x \in X$, $x \sim x$; [1%]
   (ii) (symmetric) for all $x, y \in X$, if $x \sim y$, then $y \sim x$; [2%] and
   (iii) (transitive) for all $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$. [2%]

(b) [5%] The equivalence class $[x] = \{\, y \in X : x \sim y \,\}$.

(c) [10%] We have $A = [a]$ and $B = [b]$.

- Just suppose $A \cap B$ is nonempty. Then there exists some $c \in A \cap B$. [2%]
- We show $A \subset B$. Let $x \in A$. Then $a \sim x$. [1%]
  Since also $c \in A$, then $a \sim c$. By symmetry, $c \sim a$. By transitivity, $c \sim x$. [3%]

- But $c \in B$, also, so $b \sim c$. By transitivity, $b \sim x$. This means that $x \in [b] = B$, as desired. Thus $A \subset B$. **[3%]**
- Similarly, $B \subset A$, and thus $A = B$. **[1%]** $\square$