Math 300.2 **Exam 1 Answers** **October 4, 2007**

1. (a) [6%] Let A, B, and X be sets. Then (either one of):

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$$
$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$$

   (b) [6%] A partial ordering R of X is a **total** ordering when: for each $x \in X$ and each $y \in X$, we have xRy or yRx.

   (c) [6%] If $I \subset \mathbb{N}$, if $0 \in I$, and if $(\forall n)(n \in I \Longrightarrow n+1 \in I)$, then $I = \mathbb{N}$.

2. (a) [8%] *Method 1: Use a truth table.* The truth table is as follows: [6%]

P	Q	$\neg P$	P or Q	$\neg P \,\&\, (P \text{ or } Q)$	Q	$\big(\neg P \,\&\, (P \text{ or } Q)\big) \Longrightarrow Q$
T	T	F	T	F	T	T
T	F	F	T	F	F	T
F	T	T	T	T	T	T
F	F	T	F	F	F	T

The formula *is* a tautology, because the last column in its truth table consists solely of T's. [2%]

Method 2: Use known tautologies. We have: [3%]

$$\begin{aligned}\big(\neg P \,\&\, (P \text{ or } Q)\big) &\iff \big((\neg P \,\&\, P) \text{ or } (\neg P \,\&\, Q)\big) && \text{(distributive law)}\\ &\iff (\neg P \,\&\, Q) && \text{(since } \neg P \,\&\, P \text{ is false)}\end{aligned}$$

But $(R \,\&\, Q) \Longrightarrow Q$ is a tautology so that, in particular, $(\neg P \,\&\, Q) \Longrightarrow Q$ is a tautology. [3%] Hence the given formula is a tautology. [2%]

   (b) [8%] The statement is **true**. [2%] Indeed, $(\forall n \in \mathbb{N})\,(\exists k \in \mathbb{N})\,(k > n)$ means that, for each natural number n, there is some natural number greater than n. And that is true because, for each $n \in \mathbb{N}$, we have $n+1 \in \mathbb{N}$ and $n+1 > n$. [6%]

3. Let $A = \{1, 2\}$ and $B = \{u, v, w\}$ with $u \neq v$, $v \neq w$, and $u \neq w$.

   (a) [6%] $A \times B = \{(1,u), (1,v), (1,w), (2,u), (2,v), (2,w)\}$.
Note: It would be wrong to use $\{1,u\}, \{1,v\}$, etc., as these are just doubletons—"unordered" sets—whereas elements of the cartesian product $A \times B$ are *ordered* pairs.

   (b) [6%] $\mathcal{P}(B) = \{\varnothing, \{u\}, \{v\}, \{w\}, \{u,v\}, \{u,w\}, \{v,w\}, B\}$.

   (c) [6%] By definition, $f^{-1}(\{u,v\}) = \big\{\, a \in A : f(a) \in (\{u,v\} \,\big\}$, so that

$$f^{-1}(\{u,v\}) = \{\, a \in A : f(a) = u \text{ or } f(a) = v \,\} = \{2\}.$$

Note: The inverse of a function is *not* at issue here! (Indeed, f is *not* bijective.) Rather, the expression $f^{-1}(\{u,v\})$ represents the inverse-image of the set $\{u,v\}$ under the function f.

4. [16%] Assume $A \subset B$. Certainly $A \cap B \subset A$; in fact, if $x \in A \cap B$, then $x \in A$ and $x \in B$, and so in particular $x \in A$. And $A \subset A \cap B$ because if $x \in A$, then from the assumption $A \subset B$ we have also $x \in B$ and so $x \in A \cap B$. Thus $A \cap B = A$. [8%]

Conversely, assume $A \cap B = A$. Let $x \in A$. By the assumption, $x \in A \cap B$ and so, in particular, $x \in B$. Thus $A \subset B$. [8%] $\square$

5. (a) **[6%]** One example is the function $f\colon \mathbb{Z} \to \mathbb{N}$ given by $f(n) = |n|$. [This function f is surjective because if $n \in \mathbb{N}$, then also $n \in \mathbb{Z}$ with $f(n) = n$; the function f is not injective because, for example, $f(-1) = 1 = f(1)$.]
Another example is the function $f\colon \mathbb{Z} \to \mathbb{N}$ given by:

$$f(n) = \begin{cases} 0 & \text{if } n < 0, \\ n & \text{if } n \geq 0. \end{cases}$$

[Note that the function $\mathbb{Z} \to \mathbb{N}$ given by $f(n) = n^2$ is *not* surjective, since its only values are squares! This is quite different from the case of the squaring function $\mathbb{R} \to \mathbb{R}^+$.]

(b) **[10%]** *First method:* Let $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$. Then

$$(g \circ f)(a_1) = g\big(f(a_1)\big) = g\big(f(a_2)\big) = (g \circ f)(a_2).$$

Since $g \circ f$ is injective, it follows that $a_1 = a_2$. $\square$
Second method" Let $a_1, a_2 \in A$ with $a_1 \neq a_2$. Since $g \circ f$ is injective, then

$$(g \circ f)(a_1) \neq (g \circ f)(a_2).$$

But $(g \circ f)(a_1) = g\big(f(a_1)\big)$ and $(g \circ f)(a_2) = g\big(f(a_2)\big)$, so that

$$g\big(f(a_1)\big) \neq g\big(f(a_2)\big).$$

Then $f(a_1) \neq f(a_2)$ by the functionality of g.

6. *Base step:* **[3%]** First, $2^{2\cdot 0} - 1 = 1 - 1 = 0 = 0 \cdot 3$, so that $2^{2\cdot 0} - 1$ is divisible by 3.

Inductive step: **[13%]** Now let n be a nonnegative integer and assume that $2^{2n} - 1$ is divisible by 3. **[3%]**
This assumption means there is some positive integer k for which

$$2^{2n} - 1 = 3k. \qquad \textbf{[2\%]} \tag{*}$$

We want to deduce that $2^{2(n+1)} - 1$ is divisible by 3, too. **[2%]**
Now

$$2^{2(n+1)} - 1 = 2^{2n+2} - 1 = 4 \cdot 2^{2n} - 1.$$

From (*),

$$2^{2n} = 3k + 1.$$

Then

$$\begin{aligned} 2^{2(n+1)} - 1 &= 4 \cdot 2^{2n} - 1 \\ &= 4\,(3k + 1) - 1 \\ &= 3\,(4k) + 3 \\ &= 3\,(4k + 1). \qquad \textbf{[6\%]} \end{aligned}$$

It follows that $2^{2(n+1)} - 1$ is divisible by 3, too. $\square$