Math 300.2 **Practice Exam 1 Answers** **Fall, 2007**

1. (a)

P	Q	$\neg P$	$(\neg P)$ or Q	$P \Longrightarrow Q$	$\big((\neg P) \text{ or } Q\big) \iff (P \Longrightarrow Q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

(b) The formula *is* a tautology, because the last column in its truth table consists solely of T's. In other words, because the entire formula is always true no matter what the truth values are for P and for Q.

2. (a) For each natural number, there is some (strictly) greater natural number.There is some natural number that is greater than or equal to every natural number. In other words, there is a largest natural number.

(b) You may either proceed "mechanically" or else use the meaning of the statement. To do it "mechanically", first use $\neg\,(\forall n)\big(P(n)\big) \iff (\exists)\big(\neg\,P(n)\big)$ and then use $\neg\,(\exists m)\big(Q(m)\big) \iff (\forall m)\big(\neg\,Q(m)\big)$ along with $(\neg\,m \leq n) \iff m > n$ to obtain:

$$\begin{aligned}\neg\,(\forall m \in \mathbb{N})\,(\exists n \in \mathbb{N})\,(m < n) &\iff (\exists m \in \mathbb{N})\ (\neg\,(\exists n \in \mathbb{N})\,(m < n))\\ &\iff (\exists m \in \mathbb{N})\,(\forall n \in \mathbb{N})\ \big(\neg(m < n)\big)\big)\\ &\iff (\exists m \in \mathbb{N})\,(\forall n \in \mathbb{N})\,(m \geq n)\end{aligned}$$

In words: There exists some natural number that is greater than or equal to every natural number. (More tersely: There exists a largest natural number.)

(c) Statement (a) is true, because for each $m \in \mathbb{N}$, we have $m + 1 > m$. Another reason that (a) is true is that the negated statement in (b) is false—because if there were such an m as in (b), then in particular (taking $n = m + 1$) we would have $m + 1 \leq m$, which is absurd.

3. (a) $\mathcal{P}\,(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$.

(b) $X \times Y = \{(1, z), (1, w), (2, z), (2, w), (3, z), (3, w)\}$.

(c) There are *no* injective functions from X to Y. In fact, just suppose $f\colon X \to Y$ is injective. Since $1 \neq 2$, then $f(1) \neq f(2)$. This means that either $f(1) = z$ and $f(2) = w$, or else $f(1) = w$ and $f(2) = z$. In either case, there is no possible value for $f(3)$ because if $f(3) = z$, then $f(3) = f(k)$ where either $k = 1$ or $k = 2$, contradicting that f is injective, and similarly if $f(3) = w$.

4. (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(b) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$.

See the text and/or your notes for proofs.

5. First, assume $A \subset B$. Certainly $B \subset A \cup B$ even without that assumption. And if $x \in A \cup B$, then $x \in A$ or $x \in B$; in the first case also $x \in B$ because $A \subset B$, so in either case $x \in B$. Thus $A \cup B \subset B$. Hence $A \cup B = B$.

   Conversely, assume $A \cup B = B$. If $x \in A$, then $x \in A \cup B$ and so $x \in B$. Hence $A \subset B$. $\square$

6. Relation R in X is transitive when, for all $x, y, z \in X$, if xRy and if yRz, then also xRz. In symbols (if you prefer):

$$(\forall x, y, z \in X)(xRy \ \ \& \ \ yRz \Longrightarrow xRz)$$

7. (a) Define $f\colon \mathbb{Z} \to \mathbb{Z}$ by:

$$f(n) = \begin{cases} n & \text{if } n \leq 0, \\ 0 & \text{if } n = 1, \\ n - 1 & \text{if } n \geq 2. \end{cases}$$

   The questions did not explicitly ask your to prove that your function does what it is supposed to do, but this is easy to check. Clearly this f is *not* injective because $f(1) = 0 = f(0)$. To see that f is surjective, let $k \in \mathbb{Z}$. If $k \leq 0$, then $k = f(k)$. If $k > 0$, that is, $k \geq 1$, then $k = f(k+1)$ because $k + 1 \geq 2$.

   (b) Let $z \in C$. Because $g \circ f$ is surjective, there is some $x \in A$ with $(g \circ f)(x) = z$. Then $g\big(f(x)\big) = z$. Define $y = f(x)$. Then $y \in B$, and $g(y) = g\big(f(x)\big) = z$.

8. (a) Let $x \in f^{-1}(D \cup E)$. Then $f(x) \in D \cup E$. This means $f(x) \in D$ or $f(x) \in E$. In the first case, $x \in f^{-1}(D)$; in the second case $x \in f^{-1}(E)$. Since $f^{-1}(D) \subset f^{-1}(D) \cup f^{-1}(E)$ and $f^{-1}(E) \subset f^{-1}(D) \cup f^{-1}(E)$, then in either case, $x \in f^{-1}(D) \cup f^{-1}(E)$. $\square$

   (b) We claim that also

$$f^{-1}(D) \cup f^{-1}(E) \subset f^{-1}(D \cup E)$$

   while will establish that equality in (a) *does* hold. In fact, let $x \in f^{-1}(D) \cup f^{-1}(E)$. Then $x \in f^{-1}(D)$ or $x \in f^{-1}(E)$. In the first case, $f(x) \in D$; in the second case, $f(x) \in E$. In both cases, then, $f(x) \in D \cup E$. This means $x \in f^{-1}(D \cup E)$. $\square$

9. What is to be proved, by induction on n, is that $\sum_{j=1}^{n}(2\,j-1)=n^2$.

   Base step (n = 1): First, $\sum_{j=1}^{1}(2\,j-1)=2\cdot 1-1=1=1^2$.

   Inductive step: Let $n\geq 1$ and assume

$$\sum_{j=1}^{n}(2\,j-1)=n^2.$$

   To deduce that $\sum_{j=1}^{n+1}(2\,j-1)=(n+1)^2$. We have

$$\begin{aligned}\sum_{j=1}^{n+1}(2\,j-1)&=\sum_{j=1}^{n}(2\,j-1)+\big(2\,(n+1)-1\big)\\&=n^2+2\,(n+1)-1\qquad\text{[by the inductive assumption]}\\&=n^2+2\,n+1=(n+1)^2.\quad\square\end{aligned}$$

10.

$$5^{2\,n-1}+1 \text{ is divisible by } 6.$$

   Base step (n = 1): First, $5^{2\cdot 1-1}+1=6=1\cdot 6$, so $5^{2\cdot 1-1}$ is divisible by 6.

   Inductive step: Let $n\geq 1$ and assume that

$$5^{2\,n-1}+1 \text{ is divisible by } 6.$$

   We want to deduce that $5^{2\,(n+1)-1}+1$ is divisible by 6. By the inductive assumption, there is some integer k with

$$5^{2\,n-1}+1=6\,k,$$

   and so

$$5^{2\,n-1}=6\,k-1.$$

   Then

$$\begin{aligned}5^{2\,(n+1)-1}+1=5^{2\,n+1}+1&=5^{2\,n-1}\cdot 5^2+1\\&=(6\,k-1)\cdot 25+1\\&=6\,(25\,k)-24\\&=6\,(25\,k-4).\end{aligned}$$

   But this means that $5^{2\,(n+1)-1}+1$ is divisible by 6. $\square$

11.
   - $0\in\mathbb{N}$.
   - $\sigma\colon\mathbb{N}\to\mathbb{N}$ is injective.
   - $0\notin\text{range}(\sigma)$
   - If $I\subset\mathbb{N}$, if $0\in I$, and if, for each n, $n\in I\implies\sigma(n)\in I$, then $I=\mathbb{N}$.