1. (a) [4%] First, find critical points of f [*without* expanding $f'(x)$]:

$$\begin{aligned} f'(x) = 0 &\iff x\,(x-2)^3 = 0 \\ &\iff x = 0 \text{ or } x = 2 \end{aligned}$$

Now proceed in either of two ways:

Method 1: Sample values. The derivative is continuous, so it suffices to test individual points elsewhere:

x	-1	0	1	2	3
$f'(x)$	27	0	-1	0	3
$\text{sign}(f'(x))$	$+$	0	$-$	0	$+$

Hence:

- **f is increasing for $x < 0$ and for $x > 2$**, that is, on $(-\infty, 0)$ and $(2, \infty)$; and
- **f is decreasing for $0 < x < 2$**, that is, on $(0, 2)$.

Method 2: Use inequalities. From $f'(x) = x\,(x-2)^3$:

- $x < 0 \implies x < 0$ and $x - 2 < 0 \implies x < 0$ and $(x-2)^3 < 0 \implies f'(x) > 0 \implies$ f is increasing;
- $0 < x < 2 \implies x > 0$ and $x - 2 < 0 \implies x > 0$ and $(x-2)^3 < 0$ $\implies f'(x) < 0 \implies f$ is decreasing; and
- $x > 2 \implies x > 0$ and $x - 2 > 0 \implies f'(x) > 0 \implies f$ is increasing.

(b) [4%]

$$\begin{aligned} f''(x) = (f')'(x) &= x\left(3(x-2)^2\right) + (x-2)^3(1) \\ &= (x-2)^2\left(3x + (x-2)\right) = (x-2)^2\,(4x-2) = 2(2x-1)(x-2)^2 \end{aligned}$$

Thus the critical numbers of f' are $x = 1/2$ and $x = 2$. Again, proceed in either of two ways:

Method 1: Sample values.

x	0	$1/2$	1	2	3
$f''(x)$	-8	0	2	0	10
$\text{sign}(f''(x))$	$-$	0	$+$	0	$+$

Hence:

- **f is concave down for $x < 1/2$**; and
- **f is concave up for $x > 1/2$** (or, you could say, for $1/2 < x < 2$ and for $x > 2$).

Method 2: Use inequalities. Since $2(x-2)^2 \geq 0$, the sign of $f''(x)$ is the same as that of $2x - 1$.

- $x < 1/2 \implies 2x - 1 < 0 \implies f''(x) < 0 \implies f$ is concave down;
- $1/2 < x < 2 \implies 2x - 1 > 0 \implies f''(x) > 0 \implies f$ is concave up; and
- $x > 2 \implies 2x - 1 > 0 \implies f''(x) > 0 \implies f$ is concave up.

(c) [4%] From (b), **f has only one inflection point, namely, at $x = 1/2$**.
(The concavity does *not* change at $x = 2$, so there is no inflection point there!)

(d) [4%] From (a) and the First Derivative Test, or from (a) and (b) and the Second Derivative Test:

- **f has a local maximum at $x = 0$**; and
- **f has a local minimum at $x = 2$**.

2. **[16%]** *Variables:* Let

$$t = \text{time (years) after start,}$$
$$Q(t) = \text{mass of UMa-2008 at time } t.$$

Model: For some constant k, $Q'(t) = k\,Q(t)$, that is,

$$Q(t) = Q(0)\,e^{kt}.$$

Given: $Q(2) = 0.85\,Q(0)$.

To solve for t: $Q(t) = 0.10\,Q(0)$.

Find k first: Use the given relation $Q(2) = 0.85\,Q(0)$:

$$\begin{aligned} Q(0)\,e^{k(2)} &= 0.85\,Q(0) \\ e^{2k} &= 0.85 \\ 2k &= \ln 0.85 \\ k &= \frac{\ln 0.85}{2} \qquad (\textit{Note: } k < 0) \end{aligned}$$

Then

$$Q(t) = Q(0)\,e^{\left(\frac{\ln 0.85}{2}\right)t}.$$

Finally, solve $Q(t) = 0.10\,Q(0)$ for t:

$$\begin{aligned} Q(0)\,e^{\left(\frac{\ln 0.85}{2}\right)t} &= 0.10\,Q(0) \\ e^{\left(\frac{\ln 0.85}{2}\right)t} &= 0.10 \\ \left(\frac{\ln 0.85}{2}\right)t &= \ln 0.10 \\ t &= \frac{2\,\ln 0.10}{\ln 0.85} \approx 28.336. \end{aligned}$$

Answer: About $\boxed{28.336\,\text{years}}$.

(If you interpret the question as asking how much longer it takes after the initial 2-years' decay, then the answer would be, instead, about 26.336 years.)

3. (a) **[6%]** The quotient is "0/0-form" because

$$\lim_{x \to 0} (2e^x - 2) = 2e^0 - 2 = 0, \qquad \lim_{x \to 0} x = 0.$$

Then:

$$\begin{aligned} \lim_{x \to 0} \frac{2e^x - 2}{x} &= \lim_{x \to 0} \frac{d\,(2e^x - 2)\,/dx}{d\,(x)\,/dx} && \text{(L'Hospital's Rule)} \\ &= \lim_{x \to 0} \frac{2e^x}{1} \\ &= 2e^0 = \boxed{2}. \end{aligned}$$

(b) **[5%]** L'Hospital's Rule does *not* apply here, because $\lim_{x \to 0} x + \cos x = 0 + \cos 0 = 1 \neq 0$. However, by Direct Substitution (or the ordinary Quotient Rule for limits):

$$\lim_{x \to 0} \frac{x + \sin x}{x + \cos x} = \frac{0 + \sin 0}{0 + \cos 0} = \frac{0}{1} = \boxed{0}$$

(c) **[5%]** The quotient is "∞^0-form" because

$$\lim_{x \to \infty} (1 + x) = \infty, \qquad \lim_{x \to \infty} \frac{1}{x} = 0.$$

Let

$$y = (1 + x)^{1/x}$$

so that

$$\ln y = \frac{\ln(1 + x)}{x}$$

Now

$$\lim_{x \to \infty} \ln(1 + x) = \infty, \qquad \textit{and} \lim_{x \to \infty} x = \infty,$$

so that the quotient $\ln y = \dfrac{\ln(1 + x)}{x}$ is "∞/∞-form". Then

$$\begin{aligned} \lim_{x \to \infty} (\ln y) &= \lim_{x \to \infty} \frac{\ln(1 + x)}{x} \\ &= \lim_{x \to \infty} \frac{d\,(\ln(1 + x))\,/dx}{d\,(x)\,/dx} && \text{(by L'Hospital's Rule)} \\ &= \lim_{x \to \infty} \frac{1/(1 + x)}{1} \\ &= \lim_{x \to \infty} \frac{1}{1 + x} = 0. \end{aligned}$$

Finally,

$$\begin{aligned} \lim_{x \to \infty} (1 + x)^{1/x} &= \lim_{x \to \infty} y \\ &= \lim_{x \to \infty} e^{\ln y} \\ &= e^{\lim_{x \to \infty} \ln y} \\ &= e^0 = \boxed{1} \end{aligned}$$

4. (a) **[10%]** Let $a = 0$.

$$f(x) = e^x \implies f(a) = e^0 = 1,$$
$$f'(x) = e^x \implies f'(a) = e^0 = 1$$

Then:

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= 1 + 1(x - 0) = \boxed{1 + x} \end{aligned}$$

(b) **[6%]**

$$\begin{aligned} e^{-0.2} = f(-0.2) &\approx L(-0.2) \\ &= 1 + (-0.2) = 0.8 \end{aligned}$$

This is an exact decimal, so no further rounding is needed. Thus the approximation is:

$$\boxed{e^{-0.2} \approx 0.8}$$

[*Note:* It would be misleading to write that $L(-0.2) \approx 0.8$, since the value of $L(-0.2)$ is *exactly* 0.8. And it would be *wrong* to write that $e^{-0.2} = 0.8$, since, as the TI-89 shows, $e^{-0.2} \approx 0.818731$, that is, $e^{-0.2} \approx 0.819$ when rounded to 3 decimal places.]

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5. (a) **[6%]** If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then...

$$\boxed{\textbf{there is } \textbf{\textit{some }} c \textbf{ in the open interval } (a,b) \textbf{ at which } \frac{f(b) - f(a)}{b - a} = f'(c).}$$

(b) **[10%]** The function $f(x) = \ln x$ is differentiable and continuous for $x > 0$ and hence on $[a, b] = [1, 3]$. Moreover,

$$f'(x) = \frac{1}{x}$$

for all $x > 0$. By the Mean Value Theorem, there is some c in $(1, 3)$ at which

$$\frac{f(3) - f(1)}{3 - 1} = f'(c),$$

that is,

$$\frac{\ln 3 - \ln 1}{2} = \frac{1}{c},$$

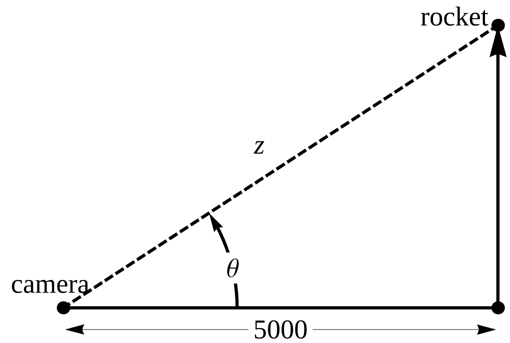
and since $\ln 1 = 0$,

$$\ln 3 = 2\,\frac{1}{c}.$$

Now $c < 3$, so that $\dfrac{1}{c} > \dfrac{1}{3}$. Hence $\ln 3 > \frac{2}{3}$.

6. **[16%]** Let

t = time (sec),
y = height (ft) of rocket at time t
θ = angle of elevation at time t of rocket from camera
z = distance (ft) at time t between camera and rocket.



Given:

$$\frac{dy}{dt} = 500.$$

To find: $\left.\frac{d\theta}{dt}\right|_{y=3000}$.

Method 1: Use the relation: $\tan\theta = \frac{y}{5000}$

Take $\frac{d}{dt}$ of the preceding relation:

$$(\sec^2\theta)\frac{d\theta}{dt} = \frac{dy/\,dt}{5000}, \qquad \text{(Chain Rule)}$$

$$(\sec^2\theta)\frac{d\theta}{dt} = \frac{500}{5000} = \frac{1}{10}, \qquad \text{(from given value of } dy/\,dt)$$

$$\frac{d\theta}{dt} = \frac{1}{10}\cos^2\theta.$$

In general,

$$\cos\theta = \frac{5000}{z}.$$

Now when $y = 3000$:

$$z = \sqrt{(5000)^2 + (3000)^2} = \sqrt{34,000,000} = 1000\sqrt{34}$$

so that

$$\cos\theta = \frac{5000}{1000\sqrt{34}} = \frac{5}{\sqrt{34}}$$

Hence

$$\left.\frac{d\theta}{dt}\right|_{y=3000} = \frac{1}{10}\left(\frac{5}{\sqrt{34}}\right)^2 = \frac{1}{10}\,\frac{25}{34} = \frac{5}{68}$$

$$\approx 0.0735294 \approx 0.074$$

Answer: The camera rotates at a rate of approximately $\boxed{0.074/\text{sec}}$, i.e., 0.074 radians/sec.

(This is approximately 4.23°/sec.)

Method 2: Use the relation $\theta = \arctan \frac{y}{5000}$.

Take $\frac{d}{dt}$:

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{1}{1+(y/5000)^2} \frac{1}{5000} \frac{dy}{dt} && \text{(Chain Rule)} \\ &= \frac{(5000)^2}{(5000)^2+y^2} \frac{1}{5000} \cdot 500 && \text{(from given value of } dy/\,dt\text{)} \\ &= \frac{1}{10} \frac{(5000)^2}{(5000)^2+y^2} \end{aligned}$$

so that

$$\left.\frac{d\theta}{dt}\right|_{y=3000} = \frac{1}{10} \frac{(5000)^2}{(5000)^2+(3000)^2} = \frac{5}{68} \approx 0.074$$