Math 131 **Exam 2 Answers** **October 22, 2008**

In #1–6, intermediate steps need not be shown
Optional ***simplified forms are shown inside {braces}***

1. **[10%]** $\dfrac{d}{d\,x}\left(x^{131}+2x^8+e^{\pi}\right)=\boxed{131x^{130}+2\cdot 8x^7+0}\quad \{=131x^{130}+16x^7\}$

2. **[10%]**

$$\begin{aligned}\frac{d}{d\,x}\left[e^{-x}\,\sin(2x)\right]&=e^{-x}\,\frac{d}{d\,x}\left[\sin(2x)\right]+(\sin 2x)\,\frac{d}{d\,x}\left(e^{-x}\right)\\&=e^{-x}\left(\cos(2x)\right)\cdot\frac{d}{d\,x}(2x)+(\sin 2x)\,e^{-x}\,\frac{d}{d\,x}(-x)\\&=\boxed{e^{-x}\left(\cos(2x)\right)\cdot(2)+(\sin 2x)\,e^{-x}\,(-1)}\quad\left\{=2e^{-x}\,\cos 2x-e^{-x}\,\sin 2x\right\}\end{aligned}$$

3. **[10%]**

$$\begin{aligned}\frac{d}{d\,x}\ln\left(e+\sqrt{x}\right)&=\frac{1}{e+\sqrt{x}}\,\frac{d}{d\,x}\left(e+\sqrt{x}\right)\\&=\boxed{\frac{1}{e+\sqrt{x}}\left(0+\frac{1}{2\sqrt{x}}\right)}\quad\left\{=\frac{1}{e+\sqrt{x}}\,\frac{1}{2\sqrt{x}}\right\}\quad\left\{=\frac{1}{2e\sqrt{x}+2x}\right\}\end{aligned}$$

4. **[10%] Method 1—quotient rule:**

$$\begin{aligned}\frac{d}{d\,x}\left(\frac{x+\sec x}{x^3+1}\right)&=\frac{\left(x^3+1\right)\dfrac{d}{d\,x}(x+\sec x)-(x+\sec x)\,\dfrac{d}{d\,x}\left(x^3+1\right)}{\left(x^3+1\right)^2}\\&=\boxed{\frac{\left(x^3+1\right)(1+\sec x\tan x)-(x+\sec x)\,(3x^2))}{\left(x^3+1\right)^2}}\\&\left\{=\frac{\left(x^3+1\right)(1+\sec x\tan x)-3x^2\,(x+\sec x))}{\left(x^3+1\right)^2}\right\}\end{aligned}$$

Method 2—logarithmic differentiation: Let

$$y=\frac{x+\sec x}{x^3+1}$$

so that

$$\ln y=\ln(x+\sec x)-\ln\left(x^3+1\right).$$

Then

$$\frac{d}{d\,x}(\ln y)=\frac{d}{d\,x}\left[\ln(x+\sec x)-\ln\left(x^3+1\right)\right],$$

that is,

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx}&=\frac{1}{x+\sec x}\cdot\frac{d}{d\,x}(x+\sec x)-\frac{1}{x^3+1}\cdot\frac{d}{d\,x}\left(x^3+1\right)\\&=\frac{1}{x+\sec x}\left(1+\tan^2 x\right)-\frac{1}{x^3+1}\left(3x^2\right)=\frac{1+\tan^2 x}{x+\sec x}-\frac{3x^2}{x^3+1}\end{aligned}$$

Finally,

$$\begin{aligned}\frac{dy}{dx}&=y\left(\frac{1+\tan^2 x}{x+\sec x}-\frac{3x^2}{x^3+1}\right)=\frac{x+\sec x}{x^3+1}\left(\frac{1+\tan^2 x}{x+\sec x}-\frac{3x^2}{x^3+1}\right)\\&=\frac{1+\tan^2 x}{x^3+1}-\frac{3x^2(x+\sec x)}{\left(x^3+1\right)^2}\end{aligned}$$

5. **[10%]**

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{x^2}-\sqrt[3]{x}\right)^{16} &= 16\left(\frac{1}{x^2}-\sqrt[3]{x}\right)^{15}\cdot\frac{d}{dx}\left(\frac{1}{x^2}-\sqrt[3]{x}\right)\\ &= 16\left(\frac{1}{x^2}-\sqrt[3]{x}\right)^{15}\cdot\frac{d}{dx}\left(x^{-2}-x^{1/3}\right)\\ &= \boxed{16\left(\frac{1}{x^2}-\sqrt[3]{x}\right)^{15}\left(-2x^{-3}-\frac{1}{3}x^{-2/3}\right)}\\ &\left\{= -16\left(\frac{1}{x^2}-\sqrt[3]{x}\right)^{15}\left(\frac{2}{x^3}+\frac{1}{3x^{2/3}}\right)\right\}\end{aligned}$$

(Logarithmic differentiation could be used instead, but it is not especially appropriate here.)

6. **[10%]**

$$\begin{aligned}\frac{d}{dx}\left(\sqrt{1-x^2}\arctan x\right) &= \sqrt{1-x^2}\cdot\frac{d}{dx}(\arctan x)+(\arctan x)\cdot\frac{d}{dx}\sqrt{1-x^2}\\ &= \sqrt{1-x^2}\,\frac{1}{1+x^2}+(\arctan x)\,\frac{1}{2\sqrt{1-x^2}}\cdot\frac{d}{dx}(1-x^2)\\ &= \boxed{\sqrt{1-x^2}\,\frac{1}{1+x^2}+(\arctan x)\,\frac{1}{2\sqrt{1-x^2}}(-2x)}\\ &\left\{\frac{\sqrt{1-x^2}}{1+x^2}-\frac{x\arctan x}{\sqrt{1-x^2}}\right\}\end{aligned}$$

7. (a) **[5%]**

$$\begin{aligned}P'(1) &= f(1)\cdot g'(1)+g(1)\cdot f'(1) && \text{(Product Rule)}\\ &= 4\times(-5)+2\times 6 && \text{(from table)}\\ &= \boxed{-8}\end{aligned}$$

(b) **[5%]**

$$\begin{aligned}H'(1) &= f'\big(g(1)\big)\cdot g'(1) && \text{(Chain Rule)}\\ &= f'(2)\cdot g'(1) && \text{(from table)}\\ &= (-1)\times(-5) && \text{(from table again)}\\ &= \boxed{5}\end{aligned}$$

8. **[10%]** The requested rate is $\dfrac{dC}{dt}$, that is $C'(t)$.

$$\begin{aligned}\frac{dC}{dt} &= K\left(e^{-at}\cdot\frac{d}{dt}(-at)-e^{-bt}\cdot\frac{d}{dt}(-bt)\right)\\ &= K\left[e^{-at}(-a)-e^{-bt}(-b)\right]\\ &= \boxed{K\left(-ae^{-at}+be^{-bt}\right)\frac{\mu\text{g/ml}}{s}}\end{aligned}$$

(The units of the derivative are just the units of C itself divided by the time unit, seconds.)

9. **[10%]** Use implicit differentiation:

$$\begin{aligned}
\frac{d}{dx}\left(y^4 + x\,y - 5x^3\right) &= \frac{d}{dx}(16) \\
4y^3 \cdot \frac{dy}{dx} + x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}\,x - 15x^2 &= 0 \\
4y^3\,\frac{dy}{dx} + x\,\frac{dy}{dx} + y \cdot (1) - 15x^2 &= 0 \\
4y^3\,\frac{dy}{dx} + x\,\frac{dy}{dx} + y - 15x^2 &= 0 \\
4y^3\,\frac{dy}{dx} + x\,\frac{dy}{dx} &= -y + 15x^2 \\
\left(4y^3 + x\right)\frac{dy}{dx} &= -y + 15x^2 \\
\frac{dy}{dx} &= \frac{-y + 15x^2}{4y^3 + x}
\end{aligned}$$

Then the slope of the tangent line to the curve at $(0, -2)$ is:

$$\begin{aligned}
\left.\frac{dy}{dx}\right|_{(x,y)=(0,-2)} &= \frac{-(-2) + 15(0)^2}{4(-2)^3 + (0)} \\
&= \frac{2}{-32} \\
&= \boxed{-\frac{1}{16}}
\end{aligned}$$

10. (a) **[2%]** $\boxed{2^y = x}$

(b) **[8%]**

$$\begin{aligned}
\frac{d}{dx}\left(2^y\right) &= \frac{d}{dx}(x) && \left[\text{take } \frac{d}{dx} \text{ of both sides in (a)}\right] \\
(\ln 2)\,2^y \cdot \frac{dy}{dx} &= 1 && \text{(by Chain rule)} \\
\frac{dy}{dx} &= \frac{1}{(\ln 2)\,2^y} \\
&= \frac{1}{(\ln 2)\,x} && \text{[from (a)]}
\end{aligned}$$

in other words,

$$\frac{d}{dx}\left(\log_2 x\right) = \boxed{\frac{1}{(\ln 2)\,x}}$$

It would be OK not to use (a) directly, omitting explicit use of y, and instead to start with

$$2^{\log_2 x} = x$$

and then proceed as follows:

$$\begin{aligned}
\frac{d}{dx}\left(2^{\log_2 x}\right) &= \frac{d}{dx}(x) \\
(\ln 2)\,2^{\log_2 x} \cdot \frac{d}{dx}\left(\log_2 x\right) &= 1 \\
(\ln 2)\,x\,\frac{d}{dx}\left(\log_2 x\right) &= 1 \\
\frac{d}{dx}\left(\log_2 x\right) &= \frac{1}{(\ln 2)\,x}
\end{aligned}$$