A. RELATIVE GRASSMANNIANS

MICHAEL FINKELBERG AND IVAN MIRKOVIĆ

Contents

Date: ?

1. **Drinfeld Grassmannii** $\mathcal{G}_{X^n}$

For a group A Grassmannii $\mathcal{G}_{X^n,A}$ are certain "rigidifications" of the stack $\mathcal{M}_A(X)$ of A-torsors on a curve X to ind-schemes. This is done in two steps: to a torsor one adds a rational section and then also an effective divisor that bounds the location of section's singularity.

Let G be a simply connected semi-simple connected algebraic group. Let $I\subseteq X_*(H_a)$ be the set of simple coroots.

1.1. **Grassmannii** $\mathcal{G}_x,\ x\in X$**.**

Let X be a smooth curve over the complex numbers. Let $x\in X$ be a closed point and denote by $\mathcal{O}_x$ the completion of the local ring at x and by $\mathcal{K}_x$ its fraction field. Then the Grassmannian $\mathcal{G}_x = G(\mathcal{K}_x)/G(\mathcal{O}_x)$ represents the following functor from C-algebras to sets :

$$R\mapsto \{\mathcal{F} \text{ a } G\text{-torsor on } X_R,\ \nu : G\times X_R^* \to \mathcal{F}|X_R^* \text{ a trivialization on } X_R^*\ \}\,.$$

Here the pairs $(\mathcal{F},\nu)$ are to be taken up to isomorphism, $X_R = X\times \operatorname{Spec}(R)$, and $X_R^* = (X-\{x\})\times\operatorname{Spec}(R)$.

Ind-scheme $\mathcal{G}_x$ depends only on the formal neighborhood of x in X.

Let us fix the isomorphism $G(\mathcal{K}_x)/G(\mathcal{O}_x)\xrightarrow{\cong}\mathcal{G}_x$. To $g\in G(\mathcal{K}_x)$ one attaches a torsor on X obtained by glueing trivial torsors G_{in} on $\hat{x}$ and G_{out} on $X-x$ by say $g\ :\ G_{in}|\tilde{x}\to G_{out}|\tilde{x}$.

1.2. **Ind-schemes** $\mathcal{G}_{X^n} = \mathcal{G}_X^{(n)}$ **over** X^n**.**

We now globalize this construction and at the same time form the Grassmannian at several points on the curve. Denote the n fold product by $X^n = X\times\cdots\times X$ and consider the functor

$$R\mapsto\{(x_1,\ldots,x_n)\in X^n(R),\ \ \mathcal{F} \text{ a } G\text{-torsor on } X_R\,,\ \nu \text{ a trivialization of } \mathcal{F} \text{ on } X_R-\cup x_i\}\ .$$

Here we think of the points $x_i : \operatorname{Spec}(R)\to X$ as subschemes of X_R by taking their graphs. One sees that the functor in (3.2) is represented by an ind-scheme $\mathcal{G}_{X^n}$.

1.3. **Locality of the Grassmannii** $\mathcal{G}_{X^n}$**.**

The ind-scheme $\mathcal{G}_{X^n}$ is obviously an ind-scheme over X^n. Its fiber $\mathcal{G}_{(x_1,\ldots,x_n)} \overset{\text{def}}{=} (\mathcal{G}_{X^n})_{(x_1,\ldots,x_n)}$ over the point $x_* = (x_1,\ldots,x_n)$ is again of local nature - restriction from X to a formal neighborhood of the support $\{x_1,...,x_n\}$ gives an identification

$$\mathcal{G}_{(x_1,\ldots,x_n)} \overset{\text{def}}{=} (\mathcal{G}_{X^n})_{(x_1,\ldots,x_n)} \xrightarrow{\cong} \prod_{y\in\{x_1,\ldots,x_n\}} (\mathcal{G}_{\hat{y}})_y.$$

The correspondence of $(\mathcal{T},(x_1,...,x_n),\tau)$ and a system of $(\mathcal{T}_y,y,\tau_y)\in(\mathcal{G}_{\hat{y}})_y,\ y\in\{x_1,...,x_n\}$; is given by:

$(\mathcal{T}_y,\tau_y) = (\mathcal{T},\tau)$ near y (i.e., on $\hat{y}$, while both are equal to the trivial torsor off $\{x_1,...,x_n\}$.

So by restriction, $(\mathcal{T}_y,\tau_y)=(\mathcal{T},\tau)|\hat{y}$, while in the opposite direction one glues $(\mathcal{T},\tau)$ from $(\mathcal{T}_y,\tau_y)$ on $\hat{y}$, $\ y\in\{x_1,...,x_n\}$; and from the trivial torsor $G{\times}X-\{x_1,...,x_n\}$, by using trivialisaztions of the pair $(\mathcal{T}_y,\tau_y)$ on $\hat{y}\cap X-\{x_1,...,x_n\}=\tilde{y}$, given by τ_y.

Because of this dependence on the formal neighborhood of $\{x_1,...,x_n\}$ (only), we will often denote $\mathcal{G}_{(x_1,...,x_n)}\stackrel{\text{def}}{=}(\mathcal{G}_{X^n})_{(x_1,...,x_n)}$, hence $\mathcal{G}_y\stackrel{\text{def}}{=}(\mathcal{G}_X)_y\cong(\mathcal{G}_{\hat{y}})_y$.

In particular, locality implies that for U open in X, restriction $\mathcal{G}_{X^n}|U^n$ is really $\mathcal{G}_{U^n}$.

1.3.1. *The precise formulation of the locality property.* For $m,n\ \in\ \mathbb{Z}$ denote by $X^{m.n}{\subseteq}X^m{\times}X^n$ the open part where the factors are disjoint. There are canonical "localization" isomorphisms $\mathcal{G}_{X^{m+n}}|X^{m,n}\ \cong\ \mathcal{G}_{X^m}{\times}\mathcal{G}_{X^n}|X^{m,n}$.

So for any disjoint $A,B{\subseteq}X$, one has an isomorphism $\mathcal{G}_{X^{m+n}}|_{A^m\times B^n}\ \cong\ \mathcal{G}_{X^m}|_{A^m}\times\mathcal{G}_{X^n}|_{B^n}$.

1.4. **Group schemes $G_{X^n}(\mathcal{O})\subseteq G_{X^n}(\mathcal{K})\supseteq G_{X^n}(\mathcal{O}_-)$.** The global analog of $G(\mathcal{O})$ is the group-scheme $G_{X^n}(\mathcal{O})$ which represents the functor

$$R\mapsto\left\{\begin{array}{l}(x_1,...,x_n)\in X^n(R),\quad \mathcal{F}\text{ the trivial }G\text{-torsor on }X_R\,,\\ \mu\text{ a trivialization of }\mathcal{F}\text{ on }\widehat{(X_R)}_{(x_1\cup...\cup x_n)}\end{array}\right\}.$$

Similarly, the global analogue of $G(\mathbb{C}[z^{-1}])$ is the group-ind-scheme $G_{X^n}(\mathcal{O}_-)$ which represents the functor

$$R\mapsto\left\{\begin{array}{l}(x_1,...,x_n)\in X^n(R),\quad \mathcal{F}\text{ the trivial }G\text{-torsor on }X_R\quad,\\ \mu\text{ a trivialization of }\mathcal{F}\text{ on }(X_R)-(x_1\cup...\cup x_n)\end{array}\right\}.$$

Finaly, the global analogue of $G(\mathcal{K}_x)$ is the group-ind-scheme $G_{X^n}(\mathcal{K})$ which represents the functor

$$R\mapsto\left\{\begin{array}{l}(x_1,...,x_n)\in X^n(R),\quad \mathcal{F}\text{ the trivial }G\text{-torsor on }X_R\,,\\ \mu\text{ a trivialization of }\mathcal{F}\text{ on }\widehat{(X_R)}-(x_1\cup...\cup x_n)\end{array}\right\}.$$

One can state it simply by

$$G_{X^n}(\mathcal{O})\,(R)\ =\ \Big\{(x_1,...,x_n,\mu),\quad(x_1,...,x_n)\in X^n(R)\quad\text{ and }\quad\mu\in G(\ \widehat{(X_R)}_{(x_1\cup...\cup x_n)}\ )\Big\},$$

$$G_{X^n}(\mathcal{O}_-)\,(R)\ =\ \{(x_1,...,x_n,\nu),\quad(x_1,...,x_n)\in X^n(R)\quad\text{ and }\quad\nu\in G(\ X_R-(x_1\cup...\cup x_n)\ )\}\,,$$

$$G_{X^n}(\mathcal{K})\,(R)\ =\ \Big\{(x_1,...,x_n,\eta),\quad(x_1,...,x_n)\in X^n(R)\quad\text{ and }\quad\eta\in G(\ \widehat{(X_R)}_{(x_1\cup...\cup x_n)}-(x_1\cup...\cup x_n)\ )\Big\}\,,$$

and the inclusions are given by restrictions.

1.4.1. In terms of these groups

$$\mathcal{G}_{X^n} \;\cong\; G_{X^n}(\mathcal{K})/G_{X^n}(\mathcal{O}).$$

The locality property of $\mathcal{G}_{X^n}$ can now be seen to come from the same property of groups $G_{X^n}(\mathcal{K})$ and $G_{X^n}(\mathcal{O})$:

$$G_{X^n}(\mathcal{K})_{(x_1,...,x_n)} \;=\; G(\; \widehat{(X_R)}_{(x_1\cup...\cup x_m)} - (x_1\cup...\cup x_m)\;) \overset{restriction}{\overset{\cong}{\to}} \prod_{y\in\{x_1,...,x_n\}} G(\; \widehat{(X_R)}_y - y\;) \;=\; \prod_{y\in\{x_1,...,x_n\}} G(\mathcal{K}_y)$$

which restricts to $G_{X^n}(\mathcal{O})_{(x_1,...,x_n)} \overset{\cong}{\to} \prod_{y\in\{x_1,...,x_n\}} G(\mathcal{O}_y)$. (However $G_{X^n}(\mathcal{O}_-)$ does not factor.)

1.5. **The fixed point set $(\mathcal{G}_{X^n})^H$ of a Cartan subgroup H.** We see that $G_{X^n}(\mathcal{K})$ acts on $\mathcal{G}_{X^n}$. In particular the constant subgroup G acts on the fibers $\mathcal{G}_{(x_1,...,x_n)} \;\cong\; \prod_{y\in\{x_1,...,x_n\}} \mathcal{G}_y$ by acting on each factor $\mathcal{G}_y$.

For any maximal torus H in G, there is a canonical identification $X_*(H) \overset{\cong}{\to} (\mathcal{G}_x)^H,\; \nu\mapsto\nu_x$. [*Description of* ν_x *in terms of* $G(\mathcal{K}_x)/G(\mathcal{O}_x)$ *and* $(T = Ind_H^G\; \nu, \tau|\hat{X}_x)$.] So

$$(\mathcal{G}_{(x_1,...,x_n)})^H \;\cong\; \prod_{y\in\{x_1,...,x_n\}} \mathcal{G}_y^H \;\cong\; \bigoplus_{y\in\{x_1,...,x_n\}} X_*(H){\cdot}y.$$

For a generic $(x_1,...,x_n)$ this is $X_*(H)^n$ and for $x_1 = \cdots = x_n$ one has only one copy $X_*(H)$.

1.5.1. **Irreducible components and the connected components of $(\mathcal{G}_{X^n})^H$.** The irreducible components of the ind-subscheme $(\mathcal{G}_{X^n})^H$ are sections $(\nu_1,...,\nu_n)_{X^n}$ of $\mathcal{G}_{X^n}\to X^n$, indexed by $(\nu_1,...,\nu_n) \in X_*(H)^n$. The value at a generic $(x_1,...,x_n) \in X^n$ is $((\nu_1)_{x_1},...,(\nu_n)_{x_n}) \in \prod_1^n\; \mathcal{G}_{x_i} = \mathcal{G}_{(x_1,...,x_n)}$. The value at any $(x_1,...,x_n) \in X^n$ lies in $\mathcal{G}_{(x_1,...,x_n)} = \prod_{y\in\{x_1,...,x_n\}} \mathcal{G}_y$ and equals $(\; (\sum_{x_i=y} \nu_i)_y\;)_{y\in\{x_1,...,x_n\}}$.

The connected components ν_{X^n} of $(\mathcal{G}_{X^n})^H$ are indexed by $\nu \in X_*(H)$: ν_{X^n} is the union of all $(\nu_1,...,\nu_n)_{X^n}$ with $\sum \nu_i = \nu$. These sections all coincide above the diagonal in X^n since the value at $(x,...,x)$ is always ν_x. For instance, for $G = SL_2$ the connected component is essentially the product of $X = \Delta_X$ and a union of $\mathbb{Z}$ lines meeting at one point.

1.5.2. *A stratification of the fixed point set.* The strata $X^K_\sim$ of the diagonal stratification of X^K are parameterized by equivalence relations $\sim$ on K. The strata of $(\mathcal{G}_{X^n})^H$ are parameterized by pairs $(\sim, \boldsymbol{\nu})$ of an equivalence class $\sim$ on $K = \{1,...,n\}$ and a map $\boldsymbol{\nu} : K/\sim\; \to X_*(H_a)$. The stratum $\boldsymbol{\nu}_{X^n_\sim}$ is a section of $\mathcal{G}_{X^n}\to X^n$ over $X^n_\sim$, the value at $(x_1,...,x_n) \in X^n_\sim$ is the family $(\boldsymbol{\nu}(y)_y)_{y\in\{x_1,...,x_n\}}$ in the fiber $(\mathcal{G}_{(x_1,...,x_n)})^H \;\cong\; \prod_{y\in\{x_1,...,x_n\}} \mathcal{G}_y^H$.

The closure of the $(\sim, \boldsymbol{\nu})$-stratum consists of the strata $(\sim', \boldsymbol{\nu}')$ where $\sim'$ is coarser then $\sim$ and $\boldsymbol{\nu}' = (K/\sim \; \to K/\sim')_* \boldsymbol{\nu}$, i.e., the value at $c' \in K/\sim'$ is $\boldsymbol{\nu}'(c') = \sum_{c \subseteq c'} \; \boldsymbol{\nu}(c)$.

1.6. **Gluing.** Let us explicate $G_{X^n}(\mathcal{K})/G_{X^n}(\mathcal{O}) \xrightarrow{\cong} \mathcal{G}_{X^n}$.

2. Stratifications of $\mathcal{G}_{X^n}$

Fix a curve $C = X$.

Let $\mathcal{G}^{(n)} = \mathcal{G}_{X^n}$ be the space classifying the triples (P, τ, d) where P is a left G-torsor over C, $d \in C^n$ and τ is a section of P off the support of d.

2.1. **Cofinite stratification by the isomorphism class of the torsor.** The projection $\mathcal{G}_{X^n} \to \mathcal{M}_G(X) \times X^n$ can be written as $G_{X^n}(\mathcal{K})/G_{X^n}(\mathcal{O}) \to G_{X^n}(\mathcal{O}_-) \backslash G_{X^n}(\mathcal{K})/G_{X^n}(\mathcal{O})$, so it simply records many ways of reconstructing torsors by gluing trivial torsors on $X - \{(x_1, ..., x_n)\}$ and $\hat{X}_{\{x_1,...,x_n\}}$.

Each of the strata, i.e., fibers of $\mathcal{G}_{X^n} \to \mathcal{M}_G(X)$, is therefore a $G_{X^n}(\mathcal{O}_-)$-torsor, and the action is given by permuting sections τ.

2.1.1. **Case $X = \mathbb{P}^1$.** The points of $\mathcal{M}_G(X)$ are indexed by $X_*(H_a)/W \ni W{\cdot}\lambda \mapsto Ind_H^G \; \lambda$. So we get strata $\mathcal{G}_{X^n}^{\lambda} = \mathcal{G}_{X^n}^{W\cdot\lambda}$, $W{\cdot}\lambda \in W \backslash X_*(H_a)$ that consist of all triples with $P \cong Ind_H^G \; \lambda$.

For $X = \mathbb{P}^1$, G_m acts on X hence on all spaces $\mathcal{G}_{X^n}$. Observe that as $G_m \ni c \to 0$, G_m-action contracts $\mathbb{A}^n$ to 0^n. So the fixed point set $(\mathcal{G}_{\mathbb{A}^n})^{G_m}$ lies in the central fiber and $(\mathcal{G}_{\mathbb{A}^n})^{G_m} = (\mathcal{G}_0)^{G_m} = G{\cdot}(\mathcal{G}_0)^H$. Its connected components are $G{\cdot}\nu_0, \; \nu \in W \backslash X_*(H)$.

2.1.2. *Lemma.* (a) As $G_m \ni c \to 0$, (i) G_m contracts $G_{X^n}(\mathcal{O}_-)$ to 1 in the fiber $G(\mathbb{C}[z^{-1}])$ at 0^n, (ii) G_m contracts $\nu_{\mathbb{A}^n}$ to $\nu_0 \in \mathcal{G}_{0^n}$.

(b) map $\mathcal{G}_{X^n} \to \mathcal{M}_G(X) \times X^n$ is G_m-equivariant and the action on the points of $\mathcal{M}_G(X)$ is trivial.

(c) The "cofinite" stratification of $\mathcal{G}_{\mathbb{A}^n}$ is the Bialnicky-Birula stratification for the G_m-action.

Proof. (i) $\mathcal{O}(\mathbb{A}^1) = \mathbb{C}[z]$ and $c \in G_m$ acts on functions by $z \mapsto c \circ z = c^{-1}{\cdot}z$, so $c \circ (z-a)^{-1} = (c^{-1}{\cdot}z - a)^{-1} = c{\cdot}(z - ca)^{-1} \to 0$.

(ii) G_m commutes with H, so it preserves irreducible components $(\nu_1, ..., \nu_n)_{X^n}$ of $(\mathcal{G}_{X^n})^H$. So, as it contracts $\mathbb{A}^n$ to 0^n, it also contracts the section $(\nu_1, ..., \nu_n)_{\mathbb{A}^n}$ to its value ν_0 at 0^n.

(c) [Messy] Clearly, ν_{X^n} lies in $\mathcal{G}_{X^n}^{W\nu}$, hence so does $\mathcal{G}_{X^n}(\mathcal{O}_-){\cdot}\nu_{X^n}$. Actually, $\mathcal{G}_{X^n}^{W\nu} = \mathcal{G}_{X^n}(\mathcal{O}_-){\cdot}\nu_{X^n}$ as the fibers of $\mathcal{G}_{X^n} \to \mathcal{M}_G(X)$ are $\mathcal{G}_{X^n}(\mathcal{O}_-)$-torsors. But (a) shows that $\mathcal{G}_{X^n}(\mathcal{O}_-){\cdot}\nu_{X^n}$ contracts to a point ν_0.

2.1.3. *Problem.* Calculate IC-stalks.

2.2. **Semi-infinite stratifications.** A choice of a Borel subgroup B associates to each triple (T,d,τ) an H_a-torsor $N\backslash\overline{B\cdot\tau}$. The semi-infinite stratification is given by this invariant. For any $\nu\in X_*(H)$ let $\mathcal{G}_{X^n}(B,\nu)$ consist of the triples (T,τ,d) with deg $\overline{B\cdot\tau}=\nu$. Here, $\overline{B\cdot\tau}\subseteq P$ is the B-reduction defined by the meromorphic section τ and the degree is the type of the H_a-torsor $(B\to H_a)_*\overline{B\cdot\tau}=N\backslash\overline{B\cdot\tau}$.

2.2.1. *$H_*(X_a)$-valued divisor $div(\tau_B)$.* The next invariant defined using B is the $X_*(H_a)$-valued divisor $div(\tau_B)$, the divisor of the section τ_B of $N\backslash\overline{B\cdot\tau}$, defined by τ.

First, we use $G_m^I\xrightarrow{\cong}H_a$ given by $I\subseteq X_*(H_a)$ to define a semigroup $\overline{H_a}\cong G_a^I$, i.e., $\mathcal{O}(H_a)=\mathbb{C}[\oplus_{i\in I}\ \mathbb{Z}\omega_i]$ contains $\mathcal{O}(\overline{H_a})=\mathbb{C}[\oplus_{i\in I}\ \mathbb{Z}_+\omega_i]$. Now a rational function f from C to H_a is a family of functions f_i and $div(f)\overset{\text{def}}{=}\sum_{i\in I}div(f_i)\cdot i\in\mathbb{Z}[I]=X_*(H_a)$.

A triple (T,τ,d) is in $\mathcal{G}_{X^n}(B,\nu)$ iff the divisor of the section τ_B has degree ν (since $\deg(\tau_B)=\deg\ div(\tau_B)$ equals $\deg(N\backslash\overline{B\cdot\tau})$). So for $n=1$, $(T,\tau,d)\in\mathcal{G}_x$ lies in $\mathcal{G}_X(B,\nu)_x$ if the order of τ_B at x equals ν, and in general, the fiber $\mathcal{G}_{X^n}(B,\nu)_{(x_1,\dots,x_n)}$ is a disjoint union of products of such strata in the ordinary Grassmannian

$$\mathcal{G}_{X^n}(B,\nu)_{(x_1,\dots,x_n)}\ \cong\ \bigsqcup_{\sum\nu_y=\nu}\ \prod_{y\in\{x_1,\dots,x_n\}}\mathcal{G}_y(B,\nu_y).$$

2.2.2. *Lemma.* (a) $[\mathcal{G}_{X^n}(B,\nu)]^H\ =\ \nu_{X^n}$.

(b) $\mathcal{G}_{X^n}(B,\nu)\ =\ N_{X^n}(\mathcal{K})\cdot\nu_{X^n}\ =\ \{p\in\mathcal{G}_{X^n},\ \lim_{G_m\ni c\to 0}\ (2\rho_{\check{\mathfrak{n}}})(c)\ \cdot p\in\nu_{X^n}\}$.

So this is a Bialnicky-Birula stratification for the action of a Cartan subgroup H.

(c) $\overline{\mathcal{G}_{X^n}(B,\nu)}\ =\ \bigcup_{\mu\leq\nu}\mathcal{G}_{X^n}(B,\mu)$, where $\leq$ (or better $\underset{B}{\leq}$) is the relation $\mu\leq\nu$ if $\nu-\mu\in\mathbb{Z}_+[\Delta_H(\mathfrak{n})]$ (opposite to the "geometric" order on characters of a Borel subgroup).

Proof. (a) An H-fixed point $(\ (\nu_y)_y\ )_{y\in\{x_1,\dots,x_n\}}\ \in(\mathcal{G}_{(x_1,\dots,x_n)})^H$, lies in $\mathcal{G}_{X^n}(B,\nu)$ iff $\deg\bar{\tau}=\sum\nu_y$ equals ν.

Now (b) and then (c) follow from the same statements for the ordinary Grassmannian, using (a) and the above decomposition of the fiber of $\mathcal{G}_{X^n}(B,\nu)$.

2.2.3. *Stratifications corresponding to opposite Borel subgroups $B_\pm$.* These are in some sense opposite stratifications. Let $B=B_+,\ H=B_+\cap B_-$ and

$$\mathcal{S}_{X^n}(\nu)=\mathcal{G}_{X^n}(B,\nu_B)\quad\text{ and }\quad\mathcal{T}_{X^n}(\nu)=\mathcal{G}_{X^n}(B_-,\nu_{B_-}),$$

here $\nu\in X_*(H)$ defines $\nu_{B_\pm}\in X_*(H_a)$ via $H\subseteq B_\pm\to H_a$. Since $B_-=w_0\cdot B_-$ the two versions $\nu_{B_\pm}\in$ are related by w_0.

Corollary. (a) $\mathcal{S}_{X^n}(\nu)$ meets $\mathcal{T}_{X^n}(\mu)$ iff $\mu\leq\nu$, i.e., iff $\mathcal{S}_{X^n}(\mu)\subseteq\overline{\mathcal{S}_{X^n}(\nu)}$.

(b) $\mathcal{S}_{X^n}(\nu)\cap\mathcal{T}_{X^n}(\nu) \;=\; \mathcal{S}_{X^n}(\nu)^H=\nu_{X^n}$, and in general

$$[\mathcal{S}_{X^n}(\nu)\cap\mathcal{T}_{X^n}(\mu)]_{(x_1,\dots,x_n)} = \bigcup_{\sum\mu_y=\mu,\sum\nu_y=\nu,\ \mu_y\le\nu_y}\ \prod_{y\in\{x_1,\dots,x_n\}} (\mathcal{S}_y)_{\mu_y}\cap(\mathcal{T}_y)_{\mu_y}.$$

Proof. One checks in each fiber $\mathcal{G}_{(x_1,\dots,x_n)}$.

2.3. **Refined semi-infinite stratifications.** This stratification is based on the invariant $(\sim,\boldsymbol{\nu})$ of a triple (T,d,τ). Point $d=(x_1,\dots,x_n)\in X^n$ lies in some stratum $X^n_\sim$ of the diagonal stratification and canonically $\{1,\dots,n\}/\sim\;\cong\{x_1,\dots,x_n\}$. The I-colored divisor $div(\tau_B)$ is the same as a function $\boldsymbol{\nu}:\{1,\dots,n\}/\sim\;\to X_*(H)$: $div(\tau_B)=\sum_{y\in\{x_1,\dots,x_n\}}\boldsymbol{\nu}(y)_B{\cdot}y$. Recall the stratification of the H-fixed points by the strata $\boldsymbol{\nu}_{X^n_\sim}$.

An H-fixed point $p=(T,d,\tau)$ with $d=(x_1,\dots,x_n)$, is of the form $p=(\boldsymbol{\nu}(y)_y)_{y\in\{x_1,\dots,x_n\}}$ for some $\boldsymbol{\nu}:\{x_1,\dots,x_n\}\to X_*(H)$. At such point $div(\tau_B)=\sum_{y\in\{x_1,\dots,x_n\}}\boldsymbol{\nu}(y)_B{\cdot}y$. Actually, $div(\tau_B)$ is $N(\mathcal{K})_{(x_1,\dots,x_n)}$-invariant, so it is constant on the orbit thru p.

2.3.1. *Lemma.* (a) Subscheme $\mathcal{G}_{X^n_\sim}(B,\boldsymbol{\nu})$ consisting of all triples with the invariant $(\sim,\boldsymbol{\nu})$ is given in the Bialnicky-Birula terms as

$$\{p\in\mathcal{G}_{X^n},\ \lim_{G_m\ni c\to 0}(2\rho_{\check{\mathfrak{n}}})(c)\cdot p\in\boldsymbol{\nu}_{X^n_\sim}\}.$$

(b) $[\mathcal{G}_{X^n_\sim}(B,\boldsymbol{\nu})]^H \;=\; \boldsymbol{\nu}_{X^n_\sim}$ and $\mathcal{G}_{X^n_\sim}(B,\boldsymbol{\nu}) \;=\; N(\mathcal{K})_{X^n_\sim}\cdot\boldsymbol{\nu}_{X^n_\sim}$.

Therefore, the invariant $div(\tau_B)$ (encoded as $\boldsymbol{\nu}$ above) of $p=(T,d,\tau)\in\mathcal{G}_{(x_1,\dots,x_n)}$, precisely describes the $N(\mathcal{K})_{(x_1,\dots,x_n)}$-orbit of p.

2.3.2. **Irreducible components of the semi-infinite strata.** The ind-subschemes of the ind-scheme $\mathcal{G}_{X^n}(B,\nu)$,

$$\mathcal{G}_{X^n}(B,\nu_1,\dots,\nu_n)\overset{\text{def}}{=}\{p\in\mathcal{G}_{X^n},\ \lim_{G_m\ni c\to 0}(2\rho_{\check{\mathfrak{n}}})(c)\cdot p\in(\nu_1,\dots,\nu_n)_{X^n}\},$$

can be thought of as "irreducible components".

Component $\mathcal{G}_{X^n}(B,\nu_1,\dots,\nu_n)$ is the closure of the stratum $\mathcal{G}_{X^n_{reg}}(B,\boldsymbol{\nu})$ that lies above the regular stratum of X^n and $\boldsymbol{\nu}(x_i)=\nu_i(x_i)$.

2.4. **Finite stratification.** Stratum $\mathcal{G}_{X^n,\lambda},\ \lambda\in W\backslash X_*(H_a)\cong W\backslash X_*(H)$, can be defined as $G_{X^n}(\mathcal{O})\cdot\lambda_{X^n}$. These strata satisfy the locality property.

2.4.1. *Lemma.* The stalk of $IC(\overline{\mathcal{G}_{X^n,\lambda}})$ at a point $(\nu_1,\dots,\nu_n)_{X^n}(x_1,\dots,x_n) = (\ (\sum_{x_i=y}\nu_i)_y\ )_{y\in\{x_1,\dots,x_n\}}$ is ...

2.5. **Relations between the strata.** We would like to extend to this setting all relations known in $\mathcal{G}$. (But for instance, $\mathcal{G}_{X^n}(B,0)\subseteq\mathcal{G}^0_{X^n}$ is not true.)

2.6. **Partial symmetrizations $\mathcal{G}_{X^{(\alpha)}}$ of $\mathcal{G}_{X^n}$.** Let $\mathcal{G}_{X^{(n)}}$ be the space that classifies the triples (T,τ,D) where T is a left G-torsor, $D\in X^{(n)}$ and τ is a section of T off the support of D. This is an ind-scheme and $\mathcal{G}_{X^n}=X^n\underset{X^{(n)}}{\times}\mathcal{G}_{X^{(n)}}$. The canonical action of the permutation group Σ_n on $\mathcal{G}_{X^n}$ is the action on the first factor X^n, hence $\mathcal{G}_{X^{(n)}}$ is the invariant theory quotient $\mathcal{G}_{X^n}//\Sigma_n$.

More generally, any map $\pi:K\to J$ with K a finite set, defines $\alpha=\sum\limits_{k\in K}\pi(k)\in\mathbb{Z}_+[J]$ and an intermediate ind-scheme $\mathcal{G}_{X^K}\to\mathcal{G}_{X^{(\alpha)}}\to\mathcal{G}_{X^{(n)}}$, for $n=|K|$. This is the invariant theory quotient

$$\mathcal{G}_{X^{(\alpha)}}\overset{\text{def}}{=}\mathcal{G}_{X^K}//\Sigma_\pi,$$

for the stabilizer Σ_π of π in Σ_K. One has $\mathcal{G}_{X^{(\alpha)}}=X^{(\alpha)}\underset{X^{(n)}}{\times}\mathcal{G}_{X^{(n)}}$ and $\mathcal{G}_{X^n}=X^n\underset{X^{(\alpha)}}{\times}\mathcal{G}_{X^{(\alpha)}}$.

It is an ind-scheme over $X^{(\alpha)}\overset{\text{def}}{=}X^K//\Sigma_\pi=\prod_{j\in J}X^{K_j}/\Sigma_{K_j}=\prod_{j\in J}X^{(k_j)}$, for $k_j=|K_j|$. We think of this as a subspace of effective J-valued divisors on X of a given degree $\sum k_j{\cdot}j=\alpha$. The fiber at $D=\sum_{j\in J}D_j{\cdot}j\in X^{(\alpha)}$ is the same as the fiber of $\mathcal{G}_{X^n}$ at any $(x_1,...,x_n)\in X^n$ above D, i.e., $\prod\limits_{y\in\text{supp}(D)}\mathcal{G}_y$.

2.6.1. **The fixed point set $(\mathcal{G}_{X^{(\alpha)}})^H$.** Since $\mathcal{G}_{X^n}\to\mathcal{G}_{X^{(\alpha)}}$ is finite $(\mathcal{G}_{X^{(\alpha)}})^H$ is the image of $(\mathcal{G}_{X^n})^H$ and the irreducible components of $(\mathcal{G}_{X^{(\alpha)}})^H$ are the images of the irreducible components of $(\mathcal{G}_{X^n})^H$. Since the parameterization $X_*(H)^K\ni\nu\mapsto\nu_{X^n}\in Irr[(\mathcal{G}_{X^n})^H]$ is Σ_K-equivariant, irreducible components of $(\mathcal{G}_{X^{(\alpha)}})^H$ are parameterized by Σ_π-orbits in $X_*(H)^K$, i.e. by $X_*(H)^{(\alpha)}$. To the orbit $\Sigma_\pi{\cdot}\boldsymbol{\nu}$ of $\boldsymbol{\nu}=(\nu_k)_{k\in K}$, there corresponds a section $\boldsymbol{\nu}_{X^{(\alpha)}}=(\Sigma_\pi{\cdot}\boldsymbol{\nu})_{X^{(\alpha)}}$ of $\mathcal{G}_{X^{(\alpha)}}\to X^{(\alpha)}$, with the value at $D=\sum_{k\in K}x_k{\cdot}\pi(k)$ equal $(\ (\sum_{x_k=y}\nu_k)_y\ )_{y\in\text{supp}(D)}\in\prod_{y\in\text{supp}(D)}\mathcal{G}_y=(\mathcal{G}_{X^{(\alpha)}})_D$. Since Σ_K preserves the connected components $\nu_{X^K}=\cup_{\sum_{k\in K}\boldsymbol{\nu}_k=\nu}\boldsymbol{\nu}_{X^K}$, their images are the connected components $\nu_{X^{(\alpha)}}=\underset{\sum_{k\in K}\boldsymbol{\nu}_k=\nu}{\cup}\boldsymbol{\nu}_{X^{(\alpha)}}$ of $(\mathcal{G}_{X^{(\alpha)}})^H$.

2.6.2. *Examples in $X_*(H_a)^{(\alpha)}$.* If $\alpha=\sum_{j\in J}\alpha_j{\cdot}j$, then $X_*(H_a)^{(\alpha)}=\prod_{j\in J}X_*(H_a)^{(\alpha_j)}$ consists of J-families $(\nu_j)_{j\in J}$ with $\nu_j=\sum_{\zeta\in X_*(H_a)}n_{j,\zeta}{\cdot}e^\zeta$ in $\mathbb{Z}_+[X_*(H_a)]$ and $\sum_{\zeta\in X_*(H_a)}n_{j,\zeta}=\alpha_j$. Map $X_*(H_a)^K\to X_*(H_a)^{(\alpha)}$ sends K-family $(\zeta_k)_{k\in K}$ to a J-family $(\sum_{\pi(k)=j}\ e^{\zeta_k})_{j\in J}$.

So, the image of $0^{(\alpha)}$ of $0^K\in X_*(H)^K$ in $X_*(H)^{(\alpha)}$ is $0^{(\alpha)}=(\alpha_j{\cdot}e^0)_{j\in J}$.

If the case $J=I$, there is a canonical map $\mathbb{Z}_+[I]\to\mathbb{Z}_+[X_*(H_a)]^I,\ \alpha\mapsto\tilde{\alpha}$. For $\alpha=\sum_{i\in I}\alpha_i{\cdot}i$ we pick an unfolding $\pi:K\to I,\ \sum_{k\in K}\pi(k)=\alpha$. It lies in $I^K\subseteq X_*(H_a)^K$ and its image in $X_*(H_a)^{(\alpha)}$ is $\tilde{\alpha}\overset{\text{def}}{=}(\sum_{\pi(k)=i}\ e^{\pi(k)})_{i\in I}\ =\ (\alpha_i{\cdot}e^i)_{i\in I}$.

2.6.3. All of the stratifications of $\mathcal{G}_{X^n}$ that we have considered, are really defined over $X^{(n)}$, i.e., they are the pull-backs of the stratifications of $\mathcal{G}_{X^{(n)}}$ for which we use similar notation. In particular, one has such stratifications of each $\mathcal{G}_{X^{(\alpha)}}$. The only difference is that the irreducible components of $\mathcal{S}_{X^{(\alpha)}}(\nu)$ (for $\nu\in X_*(H)$), are now ind-subschemes $\mathcal{S}_{X^{(\alpha)}}(\Sigma_\pi\cdot\boldsymbol{\nu}) \stackrel{\text{def}}{=} \{p\in\mathcal{G}_{X^{(\alpha)}},\ \lim_{G_m\ni c\to 0}(2\rho_{\check{\mathfrak{n}}})(c)\cdot p\in(\Sigma_\pi\cdot\boldsymbol{\nu})_{X^{(\alpha)}}\}$, indexed by $X_*(H)^{(\alpha)}$ rather then $X_*(H)^K$.

2.6.4. *Locality property.* If for $\alpha,\beta\in\mathbb{Z}_+[I]$ we denote by $X^{(\alpha,\beta)}\subseteq X^{(\alpha)}\times X^{(\beta)}$ the open part where the factors are disjoint, then there are canonical "localization" isomorphisms

$$[X^{(\alpha,\beta)}\stackrel{+}{\to}X^{(\alpha+\beta)}]^*\ \mathcal{G}_{X^{(\alpha+\beta)}}\quad\cong\quad[X^{(\alpha,\beta)}\subseteq X^{(\alpha)}\times X^{(\beta)}]^*\ \mathcal{G}_{X^{(\alpha)}}\times\mathcal{G}_{X^{(\beta)}}.$$

In particular, for U open in X, restriction $\mathcal{G}_{X^{(\alpha)}}|U^n$ is just $\mathcal{G}_{U^{(n)}}$.

2.6.5. *The diagonal stratification of* $X^{(\alpha)}$. The multi-subsets of a set S are defined as elements of some symmetric power $S^{(k)}$, we denote the image of $(s_1,...,s_k)\in S^k$ in $S^{(k)}$ by $\{\{s_1,...,s_k\}\}$. Denote by $\mathcal{P}(\alpha)$ the set of all partitions of α, i.e multi-subsets $\Gamma=\{\{\gamma_1,...,\gamma_k\}\}$ of $\mathbb{Z}_+[I]$ with $\sum_{i\in I}\gamma_i=\alpha$.

To define the *diagonal* stratification of $X^{^{(\alpha)}}$, observe that for each $D=\sum_{y\in|X|}D_y{\cdot}y\in X^{(\alpha)}$, the nontrivial D_y's form a partition of α. In this way each partition $\Gamma\in\mathcal{P}(\alpha)$ defines a stratum $X_\Gamma{}^{(\alpha)}=X_\Gamma$ and $X^{(\alpha)}=\bigsqcup_{\Gamma\in\mathcal{P}(\alpha)}X_\Gamma{}^{(\alpha)}$.

For example, the main diagonal in $X^{(\alpha)}$ is the closed stratum given by partition $\alpha=\alpha$, while the complement to all diagonals in $X^{(\alpha)}$ is the open stratum given by partition

$$\alpha=\sum_{i\in I}\underbrace{i+i+\ldots+i}_{a_i\text{ times}}.$$

3. **Restrictions**

3.1. **Parabolic subgroup P defines a "map" $\mathcal{G}_{X^n}=\mathcal{G}_{X^n,G}\to\mathcal{G}_{X^n,\bar{P}}$.** Let P be a parabolic subgroup with a unipotent radical U and the Levi group $\bar{P}=P/U$. One would like to define a map

$$r:\mathcal{G}_{X^n}=\mathcal{G}_{X^n,G}\to\mathcal{G}_{X^n,\bar{P}},\ r(T,\tau,d)=(T_P,\tau_P,d),\quad\text{by }T_P\stackrel{\text{def}}{=}U\backslash\overline{P\cdot\tau}\quad\text{and}\quad\tau_P=\text{ image of }\tau.$$

However, operation $(T,\tau,d)\mapsto\overline{P\cdot\tau}$ is only continuous on certain strata. (Is it a morphism of functors?)

One would rather try to cook up a well defined (on a subscheme) operation by modifying T_P by d.

3.1.1. More precisely, for a curve C and groups A and B, one has maps $\mathcal{G}_{C^n,A}\to\mathcal{G}_{C^n,B}$, when either (i) A maps to B, or (ii) B is a cocompact subgroup of A (but in this case the map should be defined on a subscheme of $\mathcal{G}_{C^n,A}$ only).

3.1.2. How much wrong is the claim that

$$\mathcal{G}_{X^n,G} \to \mathcal{G}_{X^n,P}$$

is an inverse to the induction given by $P \hookrightarrow G$? Map $\mathcal{G}_{X^n,P} \to \mathcal{G}_{X^n,\overline{P}}$ is a retraction with the section given by any Levi factor L of P. Therefore, the fiber of r at (S,σ,D) consists of all P-torsors Q...?

3.2. **Stratifications of $\mathcal{G}_{X^n}$ defined by P.** *This may actually work sometimes?* Any stratification of $\mathcal{G}_{X^n,\bar{P}}$ defines now a stratification of $\mathcal{G}_{X^n} = \mathcal{G}_{X^n,G}$. The basic one is by the connected components of $\mathcal{G}(\bar{P})$ (the same as the connected components of $\mathcal{G}_{X^n,\bar{P}}$): $\mathcal{G}_{X^n} = \bigcup_{\nu \in X_*[Z(\bar{P})]} \mathcal{G}_{X^n}(P,\nu)$.

It can be refined using the cofinite stratification of $\mathcal{G}_{X^n,\bar{P}}$: $\mathcal{G}_{X^n}(P)^\lambda,\ \lambda \in X_*(T)//W_L$, or the finite stratification of $\mathcal{G}_{X^n,\bar{P}}$: $\mathcal{G}_{X^n}(P)_\lambda,\ \lambda \in X_*(T)//W_L$.

4. Poisson structures on $\mathcal{G}_{\mathbb{A}^n}$

4.1. **A Poisson structure relating finite and cofinite stratifications.** In order to construct a Manin triple (imitating Drinfeld), we choose an invariant symmetric non-degenerate bilinear form κ on $\mathfrak{g}$ and a meromorphic 1-form ω on X, this gives an invariant symmetric non-degenerate bilinear form on $\mathfrak{g}_{X^n}(\mathcal{K})$ given by the sum of residues at y's: $\langle a,b \rangle \overset{\text{def}}{=} \sum_{y \in \{x_1,\dots,x_n\}} Res_y\ \kappa(a,b)\ \omega$ (should be continuous in X^n).

If we calculate $H^*(X,\mathfrak{g})$ using the affine cover of X by $X-(x_1\cup\cdots\cup x_n)$ and the formal neighborhood of $x_1\cup\cdots\cup x_n$, we get

$$0 \to \mathfrak{g} \to \mathfrak{g}_{X^n}(\mathcal{O})\ +\ \mathfrak{g}_{X^n}(\mathcal{O}_-) \to \mathfrak{g}_{X^n}(\mathcal{K}) \to \mathfrak{g} \otimes \omega_X(X) \to 0.$$

If $X=\mathbb{P}^1$ the last term is absent.

In order to make the above sum direct we choose a point $\infty \in X$ and restrict the Grassmannian to $\mathcal{G}(X-\infty)^n$. Then ∞ is disjoint from x_i's and we can define a congruence subgroup $G_{X^n}(\mathcal{O}_-)_1 = \mathrm{Ker}[G_{X^n}(\mathcal{O}_-) \xrightarrow{g \mapsto g(\infty)} G]$. Then $\mathfrak{g}_{X^n}(\mathcal{O}) \oplus \mathfrak{g}_{X^n}(\mathcal{O}_-)_1 = \mathfrak{g}_{X^n}(\mathcal{K})$ should be a Manin pair of Lie algebras over $\mathbb{A}^n$.

In order for $\mathfrak{g}_{\mathbb{A}^n}(\mathcal{O})$ to be isotropic and we choose $\omega = dx$ which is regular on $\mathbb{A}^1$. Finaly, $\mathfrak{g}_{\mathbb{A}^n}(\mathcal{O}_-)$ is isotropic since the sum of residues of a rational meromorphic form is 0.

4.1.1. The leaves are the fibers of intersections of finite strata and a modification of the cofinite strata where one replaces $G_{X^n}(\mathcal{O}_-)$ by the congruence subgroup.

4.1.2. This Poisson structure is "fiber-wise" (i.e., the fibers $\mathcal{G}_{(x_1,\dots,x_n)}$ are Poisson subspaces). The more interesting structure should involve $G^+_{X^n} \stackrel{\text{def}}{=} N_{X^n}(\mathcal{K})\cdot T_{X^n}(\mathcal{O})$, $G^-_{X^n} \stackrel{\text{def}}{=} N_{X^n}(\mathcal{K})\cdot T_{X^n}(\mathcal{O}_-)$ and something like a groupoid on X^n consisting of isomorphisms of formal neighborhoods of subschemes $x_1 \cup \cdots \cup x_n \subseteq X$ (or maybe $(x_1, ..., x_n) \in X^n$?) (something like this is needed in order to get the X^n-direction involved).

More generally, for a parabolic $P = LU$ one can use $G^+_{X^n} \stackrel{\text{def}}{=} U_{X^n}(\mathcal{K})\cdot L_{X^n}(\mathcal{O})$, $G^-_{X^n} \stackrel{\text{def}}{=} U_{X^n}(\mathcal{K})\cdot L_{X^n}(\mathcal{O}_-)$.

5. **Convolution**

5.0.3. Is there a general convolution action of $\mathcal{P}_n \stackrel{\text{def}}{=} \mathcal{P}_{G_{X^n}(\mathcal{O})}(\mathcal{G}_{X^n})$ on $\mathcal{P}(\mathcal{G}_{X^m})$ by fusion along the diagonal in $X^n \times X^m$ consisting all (x, y) with $\{x_1, ..., x_n\} \subseteq \{y_1, ..., y_m)$?

5.0.4. Or at least, does $\mathcal{P}_{G_X(\mathcal{O})}(\mathcal{G}_X)$ acts on $\mathcal{P}(\mathcal{G}_{X^n})$ by fusion along the diagonal in $X \times X^n$ consisting all $(x, (x_1, ..., x_n))$ with $x \in \{x_1, ..., x_n\}$?

5.0.5. This should in particular give the preservation of perversity result from [FM] (action on sheaves on the semi-infinite flags).