COSTELLO'S MATHEMATICAL FORMULATION OF PERTURBATIVE QUANTUM FIELD THEORY.

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Date: ?

Part 0. Intro

1. Overview

1.0.1. *QFT and mathematics.* So far, only the simplest examples of QFT have been translated into mathematics but these have had huge impact in the last quarter century. Some instances: (i) Low dimensional topology such as knot invariants, (ii) Morse theory, (iii) vertex algebras and chiral algebras of Beilinson and Drinfeld, (iv) Mirror Symmetry that relates complex geometry and symplectic geometry.

1.0.2. *Topics.* This text is an introduction to the work of Costello and Gwilliam. The first part is Costello's formulation of perturbative Quantum Field Theory (pQFT)

- (i) as mathematics,
- (ii) in large generality, and
- (iii) with ability to compute.[1]

The second part is a joint work of Costello and Gwilliam. This is an algebraic reformulation of Costello's machinery of QFT through the notion of *factorization algebras* (called *Operator Product Expansion algebras* in physics).

The goal of the text is to get acquainted with ideas of Costello-Gwilliam on factorization algebras as this technique is being developed[2] and hopefully on the level where one could actually use these techniques. The field is wide open and very important, the main question is whether these techniques can be developed into mathematical machinery that churns out theorems.

The exposition will be mathematical with comments on how/why physicists think about these ideas. The techniques of Costello and Costello-Gwilliam are elementary to start with, just a smart use of integrals. The difficulty is mostly that one has to take in several ingredients.

1.1. **Some ingredients of Costello's machine of effective QFT.** The goal is to develop the notion and construction of theories that satisfy the following two principles:

(1) *Effective action principle.* At any scale α. physics is described by an action S_α.
(2) *Locality.* In the limit as all scales are included, interactions between fields happen at points.

1.1.1. *"Effective" theories.* The starting point is a standard idea in Quantum Field Theory that

[1] This last claim is based on only one deep example that has appeared so far – construction of the Witten genus from the holomorphic Chern-Simons theory.

[2] There is no paper yet, our source is a wiki in progress.

(Wilson) *Physics depends on the scale on which it is observed.*

One very impressive consequence of this approach is that it produces an algebraic encoding of QFT by a new kind of algebras called the "factorization algebras".

1.1.2. *Feynman graphs.* The so called *Feynman integrals* are expressions which are not mathematically well defined, however they are the best known way of thinking about quantum phenomena.

Feynman graphs are a combinatorial technique for making sense of *Feynman integrals* by organizing their asymptotic expansions. For physicists, the method of Feynman graphs produces "all" numerical predictions. We would like to get some idea why this method is so useful so we will study Feynman graphs in some detail.

1.1.3. *Quantization by renormalization.* In physics, quantization means passing from a "classical" description of a system (valid on the usual scales) to a "quantum"' description. (valid on the small scales of elementary particles). Here quantization will be done by the *renormalization procedure*. Terminology *renormalization* is used in physics for ideas having to do with the change of scale. Effectively, for us renormalization is a certain game played with Feynman graphs.

We will follow Costello's approach to "quantization by renormalization" which is quite elegant. However, as stated it is not constructive so what we really get is an abstract existence statement.

1.1.4. *Factorization algebras.* This is a new algebraic structure (attributed to Beilinson-Drinfeld). Any factorization algebra lives over a topological space and the notion of factorization algebras is a new marriage of notions of algebras and sheaves.

A factorization algebra F over a topological space M associates to each open $U \subseteq M$ a vector space $F(U)$ and the multiplication $M(U) \otimes M(V) \xrightarrow{m} M(U \sqcup V)$ is defined only when U, V are disjoint.

Here are some origins of this idea. Historically, this is an incarnation of the notion of *operator product algebras* in physics. For physicist, $F(U)$ is the set of all the measurements that can be performed on U. Then the disjointness requirement comes from the *Uncertainty Principle* in Quantum Mechanics – roughly, one can only combine two measurements which are separated. The reason is that in QM each measurement influences the event while in classical physics we can observe an event without measurable impact.

The only case of operator product algebras that has been understood by mathematicians are the vertex algebras (Borcherds received Fields medal for creating the notion of vertex algebras).

Another origin of factorization algebras is the notion of E_n-algebras in topology (the homotopy theory). To start with, E_1-algebras are just the associative algebras, E_∞-algebras are the commutative associative algebras and $E_2, E_3, ...$ are in between. The point is that E_n-algebras are just the the simplest factorization algebras on the manifold $M = \mathbb{R}^n$ – the ones with a topological invariance property.

1.1.5. *Example: free theory.* So far, the only available case where factorization algebra of a QFT has been explicitly computed is the free theory when the manifold M is the real line $\mathbb{R}$. The corresponding factorization algebra is just the standard quantization of the cotangent bundle $T^*M = T^*\mathbb{R}$.

1.1.6. *BV-formalism.* The standard setting in physics is the study of the *criticality* space, also called the Euler-Lagrange space,

$$EL(S) \ = \ \{x \in \mathcal{E}; \ \ dS(x) \ = 0 \ \}$$

of a function S (called the action) on a space $\mathcal{E}$ (of *fields* of the physical theory). So, we are interested in where the differential of S vanishes.[3]

The Batalin-Vilkovisky formalism is the *derived* version of this setting, i.e., one adds the machinery of Homological Algebra in order to view the criticality locus $EL(S)$ in the setting of *derived geometry.*[4]

The standard objective of Homological Algebra is to access the "hidden" information. In the BV approach the derived geometry is used to eliminate the difficulties that appear when the space $\mathcal{F}$ (and the action S) have large symmetry.

Example. The standard setting for physicists begins with a real manifold M on which the fields live. In order to study a complex manifold X by QFT methods one views it as as a real manifold with an extra symmetry given by the motion in the ∂ directions.

1.1.7. *Example: Witten genus.* This is about a well known feature of QFT, the ease with which physicists construct *modular forms* the central objects of number theory. The difference is that they can work with elliptic curves directly while mathematicians are often forced to work with a more formal object, the modular parameter q which parametrizes elliptic curves (but there are repetitions!).

[3]The critical locus is is also studied in mathematics, for instances minimax theory in Calculus, *Morse theory* and *Milnor fiber* in Differential Geometry, *vanishing cycles* in Algebraic Geometry and Floer theory in low dimensional topology. All these are related to QFT.

[4]A higher level of derived geometry has been recently constructed by Jacob Lurie. So far it has only hit mathematics in a few places. Stay tuned.

2. **Mathematical view on Quantum Field Theory.**

For physicists, QFT is the best known approach to understanding a specific phenomenon, the elementary particles.[(5)] For mathematicians QFT does not appear as a theory of some special phenomenon but rather as a fundamental mathematical discipline (like topology, group theory,....) which influences and intertwines with many other such as geometry, topology, representation theory, homological algebra, category theory, number theory, ... and probably all the way to set theory.

We sketch the setting for QFT and its various versions. However, from the point of view of our central interests, the only important thing in this exposition is that the passage from Classical to Quantum Field theory can be viewed as shifting the interest from critical points of a function S to integrals of the form $\int\ dx\ e^{-S(x)}\phi(x)$ for various functions ϕ.

2.0.8. *Lagrangian and Hamiltonian formalism.* These are two approaches to physics. On the classical level the Lagrangian formalism is the use of the *Least Action Principle* (the system evolves so as to minimize the action functional) and its geometric home is the tangent bundle TM to the manifold M of all possible positions. The Hamiltonian formalism describes the evolution of the system in terms of the canonical symplectic structure on the cotangent bundle T^*M. The large symmetry of T^*M ("canonical transforms") allows one to pass to the coordinate system where the given problem is simple.

Historically, QM arose in the Hamiltonian setting as *quantization of the symplectic structure* on T^*M – this means that functions on T^*M were replaced by operators on the Hilbert space of L^2-functions on M. The idea is that these functions represent measurements one can perform in a given classical physical system, then for the quantum version of the system it turns out that the measurements can not be adequately represented by numbers, instead they can be viewed as self-adjoint operators on the Hilbert space. The values of a function are in this way quantized to eigenvalues of an operator and more generally to matrix coefficients of the operator.

The Lagrangian reformulation of QM (due to Dirac and Feynman) arose as a formula for the above matrix coefficients as an integral with a statistical (probabilistic) meaning. (However, the relevant probability theory is not standard – probabilities are complex numbers!)

This is the formalism we will use. While Lagrangian formalism is more general, in the Hamiltonian formalism one has a very efficient calculational tool – the Representation Theory.

2.0.9. *Classical Field Theory.* The setting is given by a manifold M, a space $\mathcal{E}$ of fields on M and a function S on fields, called *action*.

[5]This still contains many fundamental mysteries. For instance, we do not know what time is ("it arises in regions with non-turbulent gravity"). Even worse, our notion of space seems likely to break on small scales. ETC.

The fields are *local* in M ("sections of a sheaf on M") for instance they could be maps from M to some target space X. The action function describes the physical system we are interested in. It is also in some sense *local* in M The first level of the meaning of this locality property is that S is an integral over M of a quantity called the Lagrangian

$$S(x) \;=\; \int_M L(m,x), \qquad x \in \mathcal{E}.$$

(Then "locality" of S means that the Lagrangian is "a function on the jet space of fields",)

The object of interest is the *criticality* space, also called the Euler-Lagrange space,

$$EL(S) \;=\; \{x \in \mathcal{E};\;\; dS(x) \;= 0\;\} \subseteq \mathcal{E}$$

of the action function S.

The idea is that the fields x are all possible evolutions x of the system. The evolutions which are physically meaningful are those that satisfy the Equation of Motion(say $F = ma$). A particular feature of physics is that the Equation of Motion takes the form of the criticality equation $dS(x) = 0$.

2.0.10. *QFT in terms of Feynman integrals.* In physics of small scales, the initial conditions do not determine the evolution of the system. If we prepare one experiment in the same way many times, the measurement will not be the same. So, no theory can predict the numerical value of a given experiment.

What is this poor science to do? One can say that it carefully chooses a question that can be answered. What can be predicted, is the statistical behavior – the average over a large number of repetitions of the experiment. Probabilistically speaking, what theory can predict is the probability that the result of a given experiment will be certain event A (one among all possible outcomes of the experiment). In other words, the frequency with which A occurs in a large number of repetitions of the experiment.

An equivalent formulation is that for a certain measurement[6] ϕ we cannot predict the numerical value of ϕ but only the *expected value* $\langle\phi\rangle$, i.e., the average over a large number of experiments.

The probability theory says that the natural formula for the *expectation* $\langle\phi\rangle$ of ϕ is a certain average of values $\phi(x)$ when x goes over the set $\mathcal{E}$ of all possible evolutions x of the system. Here, $\phi(x)$ is the value that ϕ would take if the system would evolve according to x. This average is weighted with the probability $p(x)$ that the system will evolve according to x, so the formula should take the form of an integral

$$\langle\phi\rangle \;=\; \int_{x\in\mathcal{E}} dx\; p(x)\; \phi(x).$$

However, applying this directly to QFT does not work. The problem is that the probability $p(x)$ that we measure through repetitions of an experiment does not contain enough

[6]The following words will be synonymous for us: "experiment", measurement", "observable".

information. It turns out that the probability $p(x)$ has a refinement, the *probability amplitude* which is a complex number $\mathcal{P}(x)$ of size $|\mathcal{P}(x)|^2 = p(x)$. The point is that while the measured probability $p(x)$ does not satisfy any superposition principle (a formula for probability of a combination of experiments), $\mathcal{P}(x)$ does. In particular, the corrected formula for expected value as a superposition of contributions from all possible evolutions is physically correct:

$$\langle \phi \rangle \ = \ \int_{x \in \mathcal{E}} dx \ \mathcal{P}(x) \ \phi(x).$$

This formula is literally correct under certain normalization of the measure dx, one should require that $\langle 1 \rangle = 1$, i.e., the expected value of the constant function 1 is 1. Usually one uses a measure that has not been normalized, then $Z \stackrel{\text{def}}{=} \langle 1 \rangle$ is called the *partition function* and the true expected value of ϕ is $\langle \phi \rangle / Z$.

2.0.11. *Probability formula.* The key insight is that the probability amplitude $\mathcal{P}(x)$ is given by a classical quantity, the action S that describes the classical physical system. The formula takes form

$$\mathcal{P}(x) = \ e^{-i\frac{S(x)}{\hbar}},$$

hence

$$\langle \phi \rangle \ = \ \int_{x \in \mathcal{E}} dx \ e^{-i\frac{S(x)}{\hbar}} \ \phi(x).$$

The parameter $\hbar$ is called the *Planck scale*.

This is the first appearance of the principle that *physics depends on the scale on which it is observed.*[7] The idea is that as $\hbar \to 0$ one passes to classical physics. The mathematical machinery is the Stationary Phase Principle: as $\hbar \to 0$ the oscillations in the integrand become faster, in the limit the only contributions that survive are those from critical points of $S(x)$, i.e., from classical solutions.

The above formula relates the classical system described by action S and the quantum system. This relation involves complex numbers because QFT deals with two notions of probability. The first is the standard probability which is the way we perceive the reality as we measure the "average value of a large number of experiments". The second could be called *complex probability*, this is how the world actually functions.

We do not have any way to directly measure the phase of the complex probability $\mathcal{P}(x)$, we can only learn from it indirectly by finding how experiments interfere with each other. The basic example is Feynman's (somewhat idealized) "two slits experiment".

2.0.12. *Feynman measure.* The most obvious problem is that we do not know how to give meaning to the measure dx.[8] The only exception is Quantum Mechanics which is the case of QFT when the underlying manifold M is of dimension 1. Then $\mathcal{E}$ is a space of paths and dx is the Wiener measure on paths.

[7] In Quantum Electro Dynamics $\hbar \sim \frac{1}{137}$.

[8] Feynman thought of dx as a translation invariant measure on a vector space.

For practical purses, physicists can view this problem as largely resolved. On the theoretical level the solution is the understanding of how the (undefined) measure dx transforms, i.e., how to calculate with it. On the numerical level one makes sense of the integral by the technique of Feynman graphs (this only works in the perturbative regime, i.e., when some parameter is small).

2.0.13. *Passage to the Euclidean QFT.* The second problem is the *oscillatory nature* of the integral. Even if we knew the measure, the integrals like

$$Z \;=\; \langle 1 \rangle \;=\; \int_{x \in \mathcal{E}} dx \; e^{-i\frac{S(x)}{\hbar}}$$

could not converge absolutely since the integrand has size $|e^{-i\frac{S(x)}{\hbar}}| = 1$ and we are integrating over a large space of fields.

A way around this problem is to analytically continue the integral to its non-oscillatory form and consider

$$\langle \phi \rangle_{Euclid} \;=\; \int_{x \in \mathcal{E}} dx \; e^{-\frac{S(x)}{\hbar}} \; \phi(x).$$

Here, $e^{-\frac{S(x)}{\hbar}}$ is positive and ≤ 1 so it can be interpreted as a standard probability.

The strategy is to compute the Euclidean expectation $\langle \phi \rangle_{Euclid}$ and then analytically continue the formula to get the original expectation $\langle \phi \rangle$. This actually involves changing the metric on M from the physically natural Minkowski signature $-dt^2 + \sum_1^N \; x_i^2$ to Euclidean signature $dx_0^2 + \sum_1^N \; x_i^2$ by analytic continuation of time t to $it = x_0$ ("*Wick rotation*").

So, one calls the physically relevant version of expectation integrals (with $e^{-i\frac{S(x)}{\hbar}}$) the *Minkowski QFT*, and the the non-physical version (with $e^{-\frac{S(x)}{\hbar}}$) the *Euclidean QFT*.

Remarks. (0) This is one of a number of methods in QFT which are "not physical" in the sense that in order to get results that should agree with experiments one computes using some mathematical machinery which does not have a physical meaning. Another example are the Feynman graphs.

(1) The lack of clear foundations of QFT, i.e., the understanding of the complex valued probability and of the Feynman measure, is clearly something for mathematicians to worry about. For physicists this is not a practical problem in standard applications since physicists understand how to calculate with these objects. However, one can wonder whether proper foundations are relevant in understanding fundamentally new physics such as unification of QFT and General Relativity (through something like String Theory).

2.0.14. *Perturbative approach to Feynman integrals.* This means that we consider the case when some parameter is small and the corresponding approximations provide a simplified theory which is more manageable. This appears in several related ways, say

(1) Perturbation of Classical Field Theory to QFT: here $\hbar$ is small.
(2) Interesting theory as a perturbation of a simpler theory: here a *coupling constant* g which couples the simple action with the new ingredient is small

$$S_{interesting} \;=\; S_{simpler} + \frac{1}{g} S_{extra}.$$

(3) Perturbation ϕ of a classical solution ϕ_0 to a nearby field ϕ. Here one linearizes the problem at ϕ_0, so one replaces the original space of fields $\mathcal{E}$ with the tangent space $\mathcal{E}_{new} = \mathbb{T}_{\phi_0}\mathcal{E}$ at the classical solution. So, new fields are perturbations $e \in T_{\phi_0}\mathcal{E}$ of ϕ_0 to $\phi = \phi_0 + e$.
(4) Perturbation of the empty space. This means that the "free" action S_{free} which describes the physics of the empty space, is perturbed to a new action $S = S_{free} + I$ which describes the physics with some kind of interactions. Here, the interaction term I is considered as small.

Following Costello, we study perturbative theories such that the space of fields $\mathcal{E}$ is a vector space and 0 is a critical solution (as in (3)). Then the Taylor expansion of S starts with the quadratic term: $S = S_2 + S_{>0}$ which we call the free action $S_{free} = S_2$ and the higher terms are called the interaction term $S_{>0} = I$. The interaction term is viewed as infinitesimal (as in (4)).

Costello, actually relaxes some of these assumptions using tricks such as BV-formalism.

Our interest here is in *Feynman integrals*, a powerful tool of QFT. While these integrals are not a well defined mathematical objects, they are symbolic expressions that physicists know how to manipulate to get theoretical information and fantastically precise numerical predictions.

We will only study Feynman integrals in the *perturbative approach*, so we have to accept the restriction that certain parameter has to be small. This is the most standard way of making sense of Feynman integrals – one replaces integrals with certain divergent series which are believed to be the *asymptotic expansions* of these undefined integrals.

Part 1. Expansion of Feynman integrals according to graphs

When the space of fields is finite dimensional Feynman integrals one can make sense of Feynman integrals and calculate their expansions

$$\langle\phi\rangle \stackrel{\text{def}}{=} \int_{\mathcal{E}} dx\ e^{S(x)}\phi(x) \ = \int_{\text{graphs }\gamma} w(S,\phi;\gamma)$$

into sums of *weights* (*amplitudes*) $w(S,\phi;\gamma)$ corresponding to graphs γ (chapter 3). This graph expansion formula can be viewed as the definition of Feynman integrals in infinite dimensional situations whenever weights $w(S,\phi;\gamma)$ make sense.

The meaning of this "abstract" formalism of expansions of Feynman integrals is that it localizes on the moduli of graphs the interaction of the quadratic part g of the action and the remainder I. Here, $S = -\frac{g}{2} + I$, g is a metric on the space of fields, $-\frac{g}{2}$ is called the free action and the remainder I is called the *interaction term*.

However, in the setting of QFT the weights turn out to be themselves certain integrals which are usually infinite. So, the above "abstract" formalism of expansions of Feynman integrals is only the beggining of the story. In 4 we indicate two strategies to approximate the dual metric g^* on $\mathcal{E}^*$ by "propagators" $P \in S^2\mathcal{E}$. This results in approximation of weight integrals $w(S,\phi;\gamma)$ by new weight integrals $w_P^\gamma(I,\phi)$ which are well defined.

When one tries to restore the original weights by taking limit $P \to g^*$, infinities return. This can be cured by a choice of a "renormalized" version of the limit. The *renormalization* procedure systematically removes infinities. One measure of its subtlety is that it reveals the fact that the original Feynman integral is usually not a well defined number – one needs additional information to get rid of the choices that appear in renormalization procedure. In 2 we study renormalization in the case when it is applied to quantizing classical theories to "effective" quantum theories.

3. The abstract Feynman expansion

In this chapter we "invent" the abstract machinery of Feynman expansions by calculating Feynman integrals in the case when the space of fields is finite dimensional. Then the QFT background is not relevant and we are just constructing expansion of certain integrals on a finite dimensional vector space.

The mechanism that reduces calculation of the expansion to differentiation is presented in section 3.2 as Fourier transform. The combinatorics of this differentiation is first presented in 3.2.6 in a (misleading) generality and then repeated (the relevant version) in sections 3.3 and 3.4. The Wick formula proof in 3.3 is used as an introduction to the the general Feynman expansion 3.4. (the calculation is the same up to more combinatorics).

The combinatorics of the use of graphs is explained in categorical terms and relevant terminology is in appendices 3.6-3.8 which contain more information than is needed for the main text.

3.1. **Intro.**

3.1.1. *Perturbative approach to Feynman integrals.* Our interest here is in *Feynman integrals*, a powerful tool of QFT. While these integrals are not a well defined mathematical objects, they are symbolic expressions that physicists know how to manipulate to get theoretical information and fantastically precise numerical predictions.

We will only study Feynman integrals in the *perturbative approach*, so we have to accept the restriction that certain parameter has to be small. This is the most standard way of making sense of Feynman integrals – one replaces integrals with certain divergent series which are believed to be the *asymptotic expansions* of these undefined integrals.

3.1.2. *Feynman graphs.* The terms in this *Feynman expansion* are indexed by a certain class of graphs called *Feynman graphs*. Their role seems to be the local analysis of the interaction of two constituents (summands) of the action. The formalism applies to the case when the space of fields $\mathcal{E}$ is a vector space and one of the summands is quadratic. We call this one the "free action" S_{free}. The other summand is itself called "interaction" I (the reason for this is found in the pseudoparticle view on Feynman graphs which will appear later).

The way Feynman graphs appear in the interaction of S_{free} and I seems to be based on the fact that we know the Fourier transform of $e^{-S_{free}}$.

3.1.3. *The abstract Feynman calculus.* This is the formula for expansion of the integral over a vector space $\mathcal{E}$

$$\int_{\mathcal{E}} dx\ e^{-S(x)/\hbar}\ \phi(x)\ ,$$

as a sum over a certain class of graphs γ,

$$\sum_{\gamma}\ \frac{1}{|\mathrm{Aut}(\gamma)|} w^{\gamma}(S).$$

We will prove this formula and its generalizations when the vector space $\mathcal{E}$ is finite dimensional and the interaction part I of the action is infinitesimal. In general we will use it as the definition of the integral.

We will see that these *Feynman graphs* arise simply from differentiating polynomials by constant coefficient differential operators. So, the Feynman calculus turns out to be the combinatorics of differentiation.

On the other hand, *Feynman graphs* are really objects of category theory ("stacks") and have combinatorial nature. The categorical language that we use absorbs some traditional features of the theory – the automorphisms and colorings of graphs. For instance, in the categorical setting, a "colored graph" is an "object" while a "non-colored graph" is a

"moduli of objects".[9] The algebraic machinery that we use is really the *localization of tensor algebra* over the *moduli stacks* of *finite sets* and *graphs*.

3.1.4. *Contents.* In 3.2 we describe the setting for Feynman integrals and the ingredients for the graph expansion approach. These integrals are well defined in the finite dimensional setting and in sections 3.3 and 3.4 we calculate in the finite dimensional setting the formulas for the expansion of Feynman integrals. These formulas are then promoted in 3.9 to definitions of Feynman integrals in the infinite dimensional case.

3.2. **Ingredients: Fourier transform and combinatorics of differentiation.** We describe the setting for Feynman integrals in 3.2.1, in particular (as explained in 3.2.2), here we only consider these integrals in the finite dimensional setting. We will explain two ingredients in deriving the graph expansion of Feynman integrals: the use of Laplace transform[10] in 3.2.3 gives a formula for the expectation integral

$$\langle\phi\rangle_{g,I} \;=\; \left[\underline{e^{g^*/2}}\;(e^I\phi)\right](0)$$

in terms of differentiation by an infinite order differential operator $\underline{e^{g^*/2}}$. The graphs enter through the combinatorics of differentiation that we consider in 3.2.6.

3.2.1. *The setting for Feynman integrals.* The data $(\mathcal{E}, g, I)$ for a Feynman integral consist of a vector space $\mathcal{E}$ ("fields "), a quadratic form g on $\mathcal{E}$ and the *interaction* function I on $\mathcal{E}$.[11] We will only use the Taylor expansion of I at zero $I = \sum\, I_k$ with I_k of degree k, so we can think of I as a formal series in fields. (Sometimes $I_k = 0$ for $k < 3$.)

We call $S = -\frac{g}{2} + I$ the *action*, its free part is $S_{free} \stackrel{\text{def}}{=} -\frac{g}{2}$. The functions on $\mathcal{E}$ are called *observables*: $\mathrm{Ob} = C^\infty(\mathcal{E})$ Our goal is to define and study the Feynman (expectation) integrals of observables ϕ, with respect to the *action* $S = \frac{g(x)}{2} - I(x)$.

$$\langle\phi\rangle \;\stackrel{\text{def}}{=}\; \int_{\mathcal{E}} dx\; e^{S(x)/\hbar}\phi(x).$$

When ϕ is a product, we call $\langle\phi_1, ..., \phi_N\rangle_S \stackrel{\text{def}}{=} \langle\phi_1 \cdot \cdot \cdot \phi_N\rangle_S$ the *correlation* of functions ϕ_i.

3.2.2. *Advantages of the finite dimensional setting.* Our calculations will be done in a finite dimensional case and for a positive definite inner product. So we consider a finite dimensional vector space V over $\mathbb{R}$ with a positive definite inner product $g(x, y) = x{\cdot}y = (x, y)$. We will use the corresponding quadratic function $g(x) = g(x, x)$.[12]

[9] This is one of examples of how physicists use category theory.

[10] This is the two-sided Laplace transform, i.e., essentially the Fourier transform.

[11] In applications to QFT $\mathcal{E}$ $\mathcal{E}$ is the space of smooth sections $\mathcal{E} = C^\infty(M, E)$ of a vector bundle E over a manifold M.

[12] The case of vector spaces over a field $\Bbbk \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ is actually a special case when our vector space V has some extra structure.

The finite dimensionality allows us to choose dx as the Lebesgue measure on V normalized with respect to g so that the integral $\int_V \mathbf{dx}\, e^{-g(x)/2}$ equals one. For instance for $V = \mathbb{R}^n$, with the standard inner product and the standard measure $\boldsymbol{dx}$, one has $\int_{\mathbb{R}^n} \mathbf{dx}\, e^{-x^2/2} = \sqrt{2\pi}^n$, hence $dx = \frac{\mathbf{dx}}{(\mathbf{2\pi})^{\mathbf{n/2}}}$.

We denote by g^* the dual ("inverse") metric on V^*.

3.2.3. *Laplace transform.* We use Laplace transform in the form which takes functions on V to functions on V^*,[13]

$$(\mathcal{L}f)(J) \;=\; \int_V dx\; e^{\langle J,x\rangle} f(x), \quad J \in V^*.$$

This is just an analytic continuation of the Fourier transform, indeed, in Minkowski QFT we would use the Fourier transform rather than the Laplace transform.

Recall that a polynomial function $f \in \mathcal{O}(V) = SV^*$ on V, can be viewed as a constant coefficient differential operator $\underline{f}$ on V^*.

Lemma. (a) Laplace transform relates dual metrics g, g^* on V, V^* by

$$\mathcal{L}(e^{-g/2}) \;=\; e^{g^*/2}.$$

(b) For a polynomial function f on V

$$\mathcal{L}(f\phi) \;=\; \underline{f}(\mathcal{L}\phi).$$

(c) For a polynomial function ϕ on V and a function f on V^* such that $\mathcal{L}^{-1}f$ is defined

$$\int_V dx\; \mathcal{L}^{-1}(f)\cdot\phi \;=\; (\underline{f}\phi)(0).$$

Proof. (a) is obtained by the completion to a square. (b) is the claim the differentiation under the integral sign: $\partial_y \int_{\mathbb{R}} e^{xy} f(x)\, dx \;=\; \int_{\mathbb{R}} e^{xy}\, xf(x)\, dx$.

(c) Finally,

$$\int_V dx\; \mathcal{L}^{-1}(f)\phi \;=\; \mathcal{L}[\phi\cdot\mathcal{L}^{-1}(f)]\;(0) \;=\; [\underline{\phi}\;\mathcal{L}(\mathcal{L}^{-1}(f)](0) \;=\; [\underline{\phi}f](0) \;=\; [\underline{f}\phi](0).$$

Corollary. (a) The expectation of a polynomial function ϕ on V can be viewed in terms of applying an infinite order differential operator $\underline{e^{g^*/2}}$ and evaluating at $0 \in V$:

$$\langle\phi\rangle_{g,I} \;=\; \left[\underline{e^{g^*/2}}\;(e^I\phi)\right]\;(0).$$

(b) The free expectation of a polynomial function ϕ on V is

$$\langle\phi\rangle_{\text{free}} \;=\; \underline{\phi}(e^{g^*/2})\;(0).$$

[13]Also called the *two-sided Laplace transform*. The standard Laplace transform $\int_0^\infty dx\, e^{sx}\, e^{sx} f(x)$ is then called *one-sided Laplace transform*.

Remark. Since the pairing of SV and SV^* by

$$\langle A, B\rangle \stackrel{\text{def}}{=} \big(\underline{A}(B)\big)\ (0)$$

is symmetric, formula (b) is just a special case of (a).

Proof. (a) follows from part (c) of the lemma,

$$\langle\phi\rangle_{g,I} \ = \ \int_V dx\ e^{-\frac{g}{2}}\cdot e^I\phi \ = \ \int_V dx\ \mathcal{L}^{-1}(e^{\frac{1}{2}g^*})\cdot(e^I\phi) \ = \ \big[\underline{e^{g^*/2}}\ (e^I\phi)\big]\ (0).$$

(b) $\langle\phi\rangle_{\text{free}}$ is

$$\int_V dx\ e^{-g/2}\ \phi \ = \ \mathcal{L}(\phi e^{-g/2})\ (0) \ = \ \underline{\phi}[\mathcal{L}(e^{-g/2})]\ (0) \ = \ \underline{\phi}(e^{g^*/2})\ (0).$$

3.2.4. *Labeling graphs by tensors.* To a finite nonempty set S we attach a *flower graph* (or *star graph*) $\star_S$ which has one root and the set of prongs (petals) stemming out of this root is S. Graphs which are disjoint unions of flowers will be called flower patches.

For a homogeneous tensor $A\in S^a(V)$ the phrase "label the star graph $\star_S$ by A" (or just: "put A on $\star_S$"), will mean the following choice:

- Choose a way to write the "label" A as a sum $A=\sum_i\ v_{1,i}\cdot\cdot\cdot v_{a,i},\ v_{j,i}\in V,$ i.e., a lift of A to $\widetilde{A}=\sum_i\ v_{1,i}\otimes\cdot\cdot\cdot\otimes v_{a,i}\in V^{\otimes n}=V^{\otimes\{1,...,n\}}$.
- Choose an ordering of S.

We will loosely call the collection $v_{j,\bullet}$ the j^{th} "linear factor" of A and then we will describe the labeling by: "we have put the label $v_{1,i}\otimes\cdot\cdot\cdot\otimes v_{a,i}$ on $\star_S$" in the sense that we "label the j^{th} point in S by $v_{j,i}$".

The choices of lifting A and of ordering S will eventually be irrelevant, see .....!

By the phrase "putting a collection of tensors $F_i\in S(V_i),\ i=1,...,m$, on a graph γ" we mean that we order the set of vertices as $V_\gamma=\{v_1,...,v_m\}$ and label the flower at v_i by F_i.

3.2.5. *Contraction of tensors along a graph.* Only the first part **(A)** will be relevant for us. In **(A)** and **(C)** we generalize the contraction procedure to bigraphs just to indicate the natural scope of the technique. As we will see next, it explicates the product rule for differentiation.

A. Graphs.

B. Bigraphs. By a *bigraph with tails* we will mean a graph with tails γ with a partition of vertices $V_\gamma=V_\gamma^0\sqcup V_\gamma^1$ such that (i) all edges run between V_γ^0 and V_γ^1, and (ii) all tails stem from vertices in V_γ^1. The subclass of *bigraphs* is obtained by disallowing tails.

Let $a_\bullet=(a_1,...,a_m)\in\mathbb{N}^m$ and $b_\bullet=(b_1,...,b_n)\in\mathbb{N}^m$. By an "$(a_\bullet,b_\bullet)$-*graph with tails*" we will mean a bigraph with tails $(ga,V_\gamma^0,V_\gamma^1)$ such that (i) vertices in each of the parts are

ordered: $V_\gamma^0 = \{v_1, ..., v_m\}$ and $V_\gamma^1 = \{u_1, ..., u_n\}$, so that the valencies are given by $a_\bullet, b_\bullet$, i.e., $k_{v_i} = a_i$ and $k_{u_j} = b_j$. By "$(a_\bullet, b_\bullet)$-*graphs*" we mean the submoduli where the tails are disallowed. (Notice that these moduli are sets: the orderings prevent automorphisms.)

Consider two ordered collections of tensors $A_\bullet = (A_1, ..., A_m),\ A_i \in S(V)$, and $B_\bullet = (B_1, ..., B_n),\ B_j \in S(V^*)$, If $\deg(a_i) = a,\ \deg(B_j) = b_j$, then we can "put" $A_\bullet, B_\bullet$ on any $(a_\bullet, b_\bullet)$-graph (with tails) γ, in the sense that we put A_i and B_j on the flowers at vertices v_i^0 and v_j^1 respectively.

The "γ-contraction" $\langle A_1 \cdot\cdot A_m, B_1 \cdot\cdot B_n\rangle_\gamma$ of tensors $A_1 \cdot\cdot A_m \in SV$ and $B_1 \cdot\cdot B_n \in S(V^*)$ that have been placed on γ, is obtained by contracting the "linear factors" of $A_1 \cdot\cdot A_m$ and $B_1 \cdot\cdot B_n\rangle_\gamma$ along edges of γ. It lies in S^{t_γ} where t_γ is the number of tails of γ (these are the same as linear factors of $B_1 \cdot\cdot B_n$ which have not been contracted).

C. From graphs to bigraphs. The version of the above story that we will eventually be interested in concerns graphs rather than bigraphs. Any graph γ (possibly with tails) gives a bigraph $(\ddot{\gamma}, V^0_{\ddot{\gamma}}, V^1_{\ddot{\gamma}})$ obtained by breaking each edge e into two edges that meet at a new vertex v_e. So, vertices of $\ddot{\gamma}$ are given by vertices and edges of γ

$$V^0_{\ddot{\gamma}} = V_\gamma \quad \text{and} \quad V^1_{\ddot{\gamma}} = E_\gamma,$$

while each edge e of γ with prongs p', p'' ending in vertices v', v'' gives two edges $e_{p'}, e_{p''}$ connecting the new vertex v_e with the old vertices v', v''. The tails of $\ddot{\gamma}$ are the same as in γ.

Lemma. The bigraph contraction procedure applied to graphs yields the graph contraction procedure.

3.2.6. *Combinatorics of differentiation.* For psychological reasons we present here a graphical interpretation (i.e., in terms of graphs), of the differentiation of a product of polynomials by a product of constant coefficient differential operators. Actually, this idea will only be used in a special case where it will be described ab ovo. In this case all tensors A_i's will be the same tensor P of degree two, while tensors B_j will coincide whenever their degree are the same. This will actually change the flavor of the story since it will reduce to ordinary graphs rather than the "$(a_\bullet, b_\bullet)$-graphs" below.

We consider homogeneous tensors $A_1, ..., A_m \in\ S(V)$ and $B_1, ..., B_n \in\ S(V^*)$ of degrees $a_i = \deg(A_i)$, $b_j = \deg(B_j)$.

Lemma. (a) The derivative $(\underline{A_m} \cdot\cdot \underline{A_1})(B_1 \cdot\cdot B_n) \ \in\ S(V^*)$ localizes to $(a_\bullet, b_\bullet)$-graphs with tails,

$$(\underline{A_m} \cdot\cdot \underline{A_1})(B_1 \cdot\cdot B_n) \ = \sum_{\gamma \in\ (a_\bullet, b_\bullet)-\text{graphs with tails}} \langle A_1 \cdot\cdot A_m, B_1 \cdot\cdot B_n\rangle_\gamma.$$

(b) The pairing $\langle A_m\cdot\cdot A_1, B_1\cdot\cdot B_n\rangle \stackrel{\text{def}}{=} (\underline{A_m}\cdot\cdot\underline{A_1})(B_1\cdot\cdot B_n)\ (0) \in \Bbbk$, localizes to $(a_\bullet, b_\bullet)$-graphs,

$$\langle A_1\cdot\cdot A_m, B_1\cdot\cdot B_n\rangle \ = \sum_{\gamma\in\ (a_\bullet,b_\bullet)-\text{graphs}} \langle A_1\cdot\cdot A_m, B_1\cdot\cdot B_n\rangle_\gamma.$$

Remark. (0) While part (b) is a formality, part (a) is the content of the use of graphs.

(1) In (b) we meet the flow on fields

$$\Phi_t \stackrel{\text{def}}{=} e^{tP/2}$$

for $P = g^*$.

Proof. (a) We draw a picture consisting of two flower patches (say, each drawn as a rows of flowers) and we put tensors $A_1\cdot\cdot A_m \in S(V)$ on the upper flower patch and . $B_1\cdot\cdot B_n \in S(V^*)$ on the lower one. (So, the petals in the upper row represent copies of V and the petals in the lower row represent copies of V^*.) The upper and lower row consist of flowers $\star_{\mathcal{A}_1}, ..., \star_{\mathcal{A}_m}$ and $\star_{\mathcal{B}_1}, ..., \star_{\mathcal{B}_m}$, and we put A_i at $\star_{\mathcal{A}_i}$ and B_j at $\star_{\mathcal{B}_j}$.

The process of differentiation in the expression $(\underline{A_m}\cdot\cdot\underline{A_1})(B_1\cdot\cdot B_n)$ produces a sum of terms where each "linear factor" in $\underline{A_m}\cdot\cdot\underline{A_1}$ differentiates some "linear factor" in $B_1\cdot\cdot B_n$. These terms are parametrized by injective maps $\iota:\ \mathcal{A} = \sqcup\mathcal{A}_i \hookrightarrow \mathcal{B} = \sqcup\mathcal{B}_j$ ("the α-factor of $\underline{A_m}\cdot\cdot\underline{A_1}$ differentiates the $\iota(\alpha)$-factor of $B_1\cdot\cdot B_n$").

We can represent the ι-term by welding the petals $\alpha \in \mathcal{A}$ and $\iota(\alpha) \in \mathcal{B}$ into a segment. Then the ι-term is graphically encoded as an "$(a_\bullet, b_\bullet)$-bigraph with tails" γ.

Since the derivative $\underline{v}\lambda$ of $\lambda \in V^*$ by $v \in V$ is just the contraction $\langle v, \lambda\rangle$, the ι-summand can be described as the the "γ-contraction" $\langle A_1\cdot\cdot A_m, B_1\cdot\cdot B_n\rangle_\gamma$ of tensors $A_1\cdot\cdot A_m \in SV$ and $B_1\cdot\cdot B_n \in S(V^*)$. This gives

$$(\underline{A_m}\cdot\cdot\underline{A_1})(B_1\cdot\cdot B_n) \ = \int_{\gamma\in\ (\deg(A_\bullet),\deg(B_\bullet)-\text{bigraphs with tails}} \langle A_1\cdot\cdot A_m, B_1\cdot\cdot B_n\rangle_\gamma.$$

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(b) The evaluation at zero kills the terms for which ι is not surjective so the sum reduces to bijections ι (these terms are not affected by evaluation), i.e., to "$(\mathcal{A}, \mathcal{B})$-graphs" (we disallow tails).

$$\langle A_m\cdot\cdot A_1, B_1\cdot\cdot B_n\rangle \ = \sum_{\gamma\in\ (\mathcal{A},\mathcal{B})-\text{graphs}} \langle A_1\cdot\cdot A_m, B_1\cdot\cdot B_n\rangle_\gamma.$$

xx

Lemma. **Problem.** Let $A_1, ..., A_m \in S(V)$ and $B_1, ..., B_n \in S(V^*)$ be homogeneous tensors of degrees $a_i = \deg(A_i)$, $b_j = \deg(B_j)$.

(a) We consider two calculations:

(1) The derivative $(\underline{A_m}\cdots\underline{A_1})(B_1\cdots B_n) \in S(V^*)$.

(2) The pairing $\langle A_m\cdots A_1, B_1\cdots B_n\rangle \stackrel{\text{def}}{=} (\underline{A_m}\cdots\underline{A_1})(B_1\cdots B_n)\,(0) \in \mathbb{R}$.

A graphical presentation of calculations. (1) We draw a picture consisting of two rows of flowers, the upper row represents $B_1\cdots B_n \in S(V^*)$ and the lower row represents $A_1\cdots A_m \in S(V)$. (The petals in the upper row represent copies of V^* and the petals in the lower row represent copies of V.) The upper row consists of flowers $\star_{\mathcal{A}_1}, ..., \star_{\mathcal{A}_m}$ of sizes $|\mathcal{A}_i| = a_i$ and the lower row of flowers $\star_{\mathcal{B}_1}, ..., \star_{\mathcal{B}_m}$ of sizes $|\mathcal{B}_j| = b_j$, so that we can label $\star_{\mathcal{A}_i}$ with the tensor A_i and $\star_{\mathcal{B}_j}$ with B_j.

The process of differentiation in $(\underline{A_m}\cdots\underline{A_1})(B_1\cdots B_n)$ produces a sum of terms where each "linear factor" in $\underline{A_m}\cdots\underline{A_1}$ differentiates some "linear factor" in $B_1\cdots B_n$. These terms are parametrized by injective maps $\iota : \mathcal{A} = \sqcup\mathcal{A}_i \to \mathcal{B} = \sqcup\mathcal{B}_j$ ("the α-factor of $\underline{A_m}\cdots\underline{A_1}$ differentiates the $\iota(\alpha)$-factor of $B_1\cdots B_n$").

We can represent the ι term by welding the petals $\alpha \in \mathcal{A}$ and $\iota(\alpha) \in \mathcal{B}$ into a segment. So, ι is graphically encoded as an "$(\mathcal{A},\mathcal{B})$-graph with tails" γ, where the phrase means a graph γ which consists of (i) vertices which are partitioned into two subsets $\mathcal{A}$ and $\mathcal{B}$, (ii) edges that go between $\mathcal{A}$ and $\mathcal{B}$, and (ii) tails (=petals=prongs) stemming from vertices in $\mathcal{B}$.

Since the derivative $\underline{v}\lambda$ of $\lambda \in V^*$ by $v \in V$ is just the contraction $\langle v, \lambda\rangle$, the summand can be described as the the "γ-contraction" $\langle A_1\cdots A_m, B_1\cdots B_n\rangle_\gamma$ of tensors $A_1\cdots A_m \in SV$ and $B_1\cdots B_n \in S(V^*)$. So,

$$(\underline{A_m}\cdots\underline{A_1})(B_1\cdots B_n) \;=\; \sum_{\gamma\in\ (\mathcal{A},\mathcal{B})-\text{graphs with tails}} \langle A_1\cdots A_m, B_1\cdots B_n\rangle_\gamma.$$

(b) The evaluation at zero kills the terms for which ι is not surjective so the sum reduces to bijections ι (these terms are not affected by evaluation), i.e., to "$(\mathcal{A},\mathcal{B})$-graphs" (we disallow tails).

$$\langle A_m\cdots A_1, B_1\cdots B_n\rangle \;=\; \sum_{\gamma\in\ (\mathcal{A},\mathcal{B})-\text{graphs}} \langle A_1\cdots A_m, B_1\cdots B_n\rangle_\gamma.$$

yy

A. The case when all A_i are of degree 2.

B. The case when all A_i are the same and of degree 2.

3.3. **Feynman integrals in the case of free action (Wick formula).** The Wick formula is a special case of the Feynman expansion formula that we cover in the next section 3.4. The general case is a perturbation of the special case by adding an infinitesimally

small "interaction" term in the action. We use the same proofs for both, so this section can be viewed as a warm up (with less combinatorics) for the next one.[14]

3.3.1. *Wick's theorem.* It calculates correlators of linear functions for the "*free*" action $S = -g/2$.

By a *pairing* on a set A we mean a partition of A into pairs of distinct elements. So, a partition of A is a family of disjoint two element subsets $\Gamma = \big\{\{p_1, q_1\}, ..., \{p_M, q_M\}\big\}$ that covers A. However, no order is chosen on the pairs nor on the of elements of any pair. We denote by Pai_N the set of all pairings Γ of the set $\{1, ..., N\}$

Theorem. The free correlator of linear functionals $\phi_i \in V^*$ is the sum of all products of "contractions" $\mathsf{g}(\phi_i, \phi_j)$ of pairs of functionals with respect to the "inverse" inner product g on V^* :

$$\langle \phi_1, ..., \phi_N \rangle_{\text{free}} \;=\; \sum_{\Gamma \in \mathsf{Pai}_N} \; \prod_{\{i,j\} \in \Gamma} g^*(\phi_i, \phi_j).$$

Our proof will use the Laplace transform.

3.3.2. *A proof of Wick formula.* We know that $\langle \phi \rangle_{\text{free}}$ is the derivative

$$\underline{\phi}(e^{g^*/2})\;(0) \;=\; \sum_{M=0}^{\infty} \; \underline{\phi_N} \cdot \cdot \underline{\phi_1} \frac{(g^*/2)^M}{M!}\;(0).$$

If N is odd this is zero and the same is true for the RHS since $\mathsf{Pai}_N = \emptyset$. So let N be even, then all terms are zero except for $M = N/2$, so

$$\langle \phi \rangle_{\text{free}} \;=\; \underline{\phi_N} \cdot \cdot \underline{\phi_1} \frac{(g^*/2)^M}{M!}\;(0) \;=\; \frac{1}{2^M M!}\;\; \underline{\phi_N} \cdot \cdot \underline{\phi_1}\; {g^*}^M.$$

The differentiation $\underline{\phi_N} \cdot \cdot \underline{\phi_1}\; {g^*}^M$ produces $N!$ terms where N factors of $\underline{\phi_1} \cdot \cdot \underline{\phi_1}$ differentiate N-factors of ${g^*}^M$ (g^* is a sum of terms which are products of two linear factors!). Since the double derivative $\underline{\phi_{p_j} \phi_{q_j}}\; g^*$ of the metric g^* is the contraction $g^*(\phi_{p_j}, \phi_{q_j})$ of the metric g^* with $\phi_{p_j} \otimes \phi_{q_j}$, these are of the form

$$\underline{\phi_{p_1} \phi_{q_1}}\; g^* \cdot \cdot \cdot \; \underline{\phi_{p_M} \phi_{q_M}}\; g^* \;=\; g^*(\phi_{p_1}, \phi_{q_1}) \cdot \cdot g^*(\phi_{p_M}, \phi_{q_M})$$

where $(p_1, ..., p_M, q_1, ..., q_M)$ is a permutation of $(1, ..., N)$.

Such summand defines a partition of indices $1, ..., N$ into pairs p_j, q_j, we say that the collection of pairs $\Gamma = \big\{\{p_1, q_1\}, ..., \{p_M, q_M\}\big\}$ is a *pairing* on the set $\{1, .., N\}$. The value of the summand $g^*(\phi_{p_1}, \phi_{q_1}) \cdot \cdot g^*(\phi_{p_1}, \phi_{q_1})$ can be written as $\prod_{\{p,q\} \in \Gamma}\; g^*(\phi_p, \phi_q\}$. So it only depends on the pairing Γ but not on the full data of a permutation $(p_1, ..., p_M, q_1, ..., q_M)$.

[14] Alternatively, one could deduce the Feynman expansion by applying Wick formula infinitely many times.

The multiple ways to get the same pairing Γ from different permutations $(p_1, ..., p_M, q_1, ..., q_M)$ introduce repetitions in the sum for the derivative $(\underline{\phi_1} \cdot \cdot \cdot \underline{\phi_1})g^{*M}$. In order to upgrade a pairing Γ to a permutation $(p_1, ..., p_M, q_1, ..., q_M)$, we need to choose for each of the M pairs $\{p, q\}$ in Γ, which of the M-factors in g^{*M} it will differentiate, and also whether $\underline{\phi_p}$ will differentiate the first or the second factor of the quadratic expression g^*. This gives 2^M choices, hence

$$(\underline{\phi_1} \cdot \cdot \cdot \underline{\phi_1})g^{*M} \;=\; 2^M M! \sum_{\Gamma \in \mathsf{Pai}_N} \prod_{\{p,q\}\in\Gamma} g^*(\phi_p, \phi_q\}.$$

3.3.3. *A graphical interpretation of the calculation.* Graphically, the pairing Γ is represented by a simple graph consisting of M segments that join pairs of points p, q that lie Γ. We will now go once again through the proof 3.3.2 of Wick formula, in order to formulate how this graph (i.e., the corresponding summand of $\langle\phi\rangle_{\text{free}}$) arises through the process of differentiation. This is how Feynman graphs will arise.

We draw a picture consisting of two rows of N points each. The points in the upper row represent copies of V^* and the points in the lower row represent copies of V.

The points in the upper row are labeled by $1, ..., N$ and on the point i we "put" $\phi_i \in V^*$. Then the first row represents the product of linear functions $\phi = \phi_1 \cdot \cdot \cdot \phi_N$ which, when viewed as a tensor, is an element of $V \otimes \cdot \cdot \cdot \otimes V = V^{\otimes n}$.

The points in the lower row are joined into $M = N/2$ segments, each representing a copy of S^2V. On each segment $[a_j, b_j]$ we "put" one copy of $g^* \in S^2V$. More precisely, the label $g^* = \sum\, v'_k \otimes v''_k$ is a sum of labels $v_k \otimes v'_k$ and we "put" v'_k on the point a_j and v''_k on b_j. (Which point is the first does not matter because g^* is a symmetric tensor!) Now, the second row represents the product the element g^{*M} of $(S^2V)^{\otimes M}$.

The process of differentiating g^{*M} by $\underline{\phi_1} \cdot \cdot \cdot \underline{\phi_1}$ produces $N!$ terms where N factors of $\underline{\phi_1} \cdot \cdot \cdot \underline{\phi_1}$ differentiate N-factors of g^{*M} (g^* is a sum of terms which are products of two factors!). Since the double derivative $\underline{\phi_{p_1}\phi_{q_1}}g^*$ of the metric g^* is the contraction $g^*(\phi_{p_1}, \phi_{q_1})$ of the metric g^* with $\phi_{p_1} \otimes \phi_{q_1}$, these are of the form

$$\underline{\phi_{p_1}\phi_{q_1}}g^* \cdot \underline{\phi_{p_M}\phi_{q_M}}g^* \;=\; g^*(\phi_{p_1}, \phi_{q_1}) \cdot \cdot \cdot g^*(\phi_{p_1}, \phi_{q_1})$$

where $p_1, ..., p_M, q_1, ..., q_M$ is a permutation of $1, ..., N$. We can represent such term by joining the points a_j, b_j in the j^{th} segment respectively with the points p_j, q_j in the upper row, by means of segments.

Notice that now the points $1, ..., N$ are joined in pairs p_j, q_j by segments (each of which consists of three pieces). We say that the collection of pairs $\Gamma = \big\{\{p_1, q_1\}, ..., \{p_M, q_M\}\big\}$ is a *pairing* on the set $\{1, .., N\}$. The value of the summand $g^*(\phi_{p_1}, \phi_{q_1}) \cdot \cdot \cdot g^*(\phi_{p_1}, \phi_{q_1})$ can be written as $\prod_{\{p,q\}\in\Gamma}\; g^*(\phi_p, \phi_q\}$. So the value of the summand only depends on how we paired the points in the upper row but not at all on how we factored this pairing through the lower row – it does not require the full data of a permutation $(p_1, ..., p_M, q_1, ..., q_M)$.

Multiple ways to get the same pairing Γ from different permutations $(p_1, ..., p_M, q_1, ..., q_M$, i.e., to factor the pairing in the upper row through points in the lower row, introduce repetitions in the sum for the derivative $(\underline{\phi_1}\cdot\cdot\cdot\underline{\phi_1})g^{*M}$. In order to factor a pairing Γ through the lower row we need to choose for each of the M pairs $\{p, q\}$ in Γ one of M segments $[a_j, b_j]$ in the lower row ($M!$ choices) and also to choose whether p will be connected to a_j or b_j (2^M choices). Therefore,

$$(\underline{\phi_1}\cdot\cdot\cdot\underline{\phi_1})g^{*M} \;=\; 2^M M! \sum_{\Gamma\in \mathsf{Pai}_N} \prod_{\{p,q\}\in\Gamma} g^*(\phi_p, \phi_q\}.$$

Therefore, if we denote by Pai_N the set of all pairings Γ of the set $\{1, ..., N\}$

$$(\underline{\phi_1}\cdot\cdot\cdot\underline{\phi_1})g^{*M} \;=\; 2^M M! \sum_{\Gamma\in \mathsf{Pai}_N} \prod_{\{p,q\}\in\Gamma} g^*(\phi_p, \phi_q\}.$$

3.4. **Feynman expansion of expectation integrals.** Let $\mathcal{G}_N$ be the moduli of graphs with N tails colored $1, ..., N$ (see the appendix 3.6.1). By an integral $\int_{\mathcal{C}\ni c} w(c)$ over a category $\mathcal{C}$ we mean the sum over the set $\pi_0(\mathcal{C})$ of isomorphism classes $[c]$ of objects c in $\mathcal{C}$, weighed using the automorphisms of c

$$\int_{c\in\mathcal{C}} w(c) \;\stackrel{\text{def}}{=}\; \sum_{[c]\in\pi_0(\mathcal{C})} \frac{1}{|\mathrm{Aut}_{\mathcal{C}}(c)|}.$$

The "inverse" inner product g^* on V^* lies in S^2V. We use terminology *propagator* for any symmetric degree two tensors $P \in S^2V$.

3.4.1. (γ, P)*-amplitude* $w_P^\gamma(I)$. For a graph $\gamma \in \mathcal{G}_N$ and any propagator P, i.e., any element $\mathsf{P} \in S^2V$, we define

$$w_{\mathsf{P}}^\gamma(I, \phi) \stackrel{\text{def}}{=} \langle \mathsf{P}_\gamma, (I, \phi)_\gamma\rangle$$

as the γ-contraction of tensors $\mathsf{P}_\gamma \in SV$ and $(I, \phi)_\gamma \in S(V^*)$ which is defined in the following way.

We put the label P on each edge of the graph γ, we put $\phi_i \in V^*$ on the vertex of γ labeled by i and $I_k \in S^k(V^*)$ on the unlabeled k-valent vertices. Then P_γ is the tensor product over edges $e \in E(\gamma)$: $P_\gamma = \gamma^{E_\gamma}$ while $(I, \phi)_\gamma$ is the tensor product over vertices: $(I, \phi)_\gamma = \otimes_1^N \phi_i \otimes \otimes_{v\in V(\gamma)} I_{k(v)}$, where $V(\gamma)$ is the set of unlabeled vertices and $k(v)$ is the valency of the unlabeled vertex v.

Again, the precise meaning of labeling is that the labels I_k are sums of labels of the form $\alpha_1\otimes\cdot\cdot\cdot\alpha_k$ with $\alpha_i \in V^*$, and we place each factor α_i at one petal of the flower. Because I_k is a symmetric tensor it does not matter how we order the petals. Similarly g^* is a sum of terms $a_1\otimes a_2$ with $a_i \in V$ and we place a_i at the i^{th} boundary point of the segment.

In this way, each prong p of γ carries two kinds of labels: a vector in V from the edge that p lies on and a linear functional in V^* from the vertex the prong starts at. The

γ-contraction means that we contract these "linear factors" of P_γ and $(I,\phi)_\gamma$ along the prongs of γ.

3.4.2. *Theorem.* Assume that the interaction I is infinitesimal. The correlator of linear functionals $\phi_1, ..., \phi_N \in V^*$

$$\langle \phi_1, ..., \phi_N \rangle_{-\frac{g}{2}+I} \stackrel{\text{def}}{=} \int_V dx\ e^{-\frac{g}{2}(x)+I(x)}\ \phi_1(x)\cdot\cdot\phi_N(x)$$

can be viewed as

(a) [Feynman expansion.] The integral over the moduli of graphs $\mathcal{G}_N$ of (γ, g^*)-weights $w^\gamma_{g^*}(I,\phi)$ of the pair (I,ϕ) :

$$\int_{\Gamma\in\mathcal{G}_N} w^\gamma_P(I,\phi).$$

(b) The pairing (by differentiating and evaluating at zero) of the infinite order differential operator $e^{-\frac{1}{2}g^*\underline{\ \ }}$ and the formal function $e^I\phi$

$$e^{-\frac{1}{2}g^*\underline{\ \ }}(e^I\phi)\ (0).$$

Remark. The infinitesimality assumption means that $I_k = g_k\widetilde{I}_k$ with $I_k \in S^k(V^*)$ and g_k's some infinitesimal parameters. Then e^I is a a formal power series in V^* and $g_0, g_1, ...$, and the integral is well defined as an element of $\mathbb{C}[[g_0, g_1, ...]]$ by integrating the coefficients of powers $g^{\vec{n}}$ – these integrals are the free correlators calculated by the Wick formula.

So, the only problem is to organize the resulting huge sum in a reasonable way. This is achieved as in the proof of the Wick formula, except that the class of graphs that appear is $\mathcal{G}_N$ rather than Pai_N.

3.4.3. *A proof of the Feynman expansion up to constants v_γ.* Let $\phi = \prod_{i=1}^N \phi_i$. The proof is the same as for Wick formula, except that this time we differentiate by $\underline{e^I\phi}$ rather than just by $\underline{\phi}$. In the manipulation

$$\langle\phi\rangle_{-\frac{g}{2}+I}\int_V dx\ e^{-\frac{g}{2}}e^I\phi\ =\ \mathcal{L}(e^I\phi\ e^{-\frac{g}{2}})\ (0)\ \stackrel{(\star)}{=}\ [\underline{e^I\phi}(\mathcal{L}e^{-\frac{g}{2}})]\ (0) =\ [\underline{e^I\phi}(e^{-g^*/2})]\ (0)\ ,$$

the step $(\star)$ is again the lemma 3.2.3.b – it applies because $e^I\phi$ is a formal sum of polynomials.

Now, part (a) is clear since the pairing of $A \in SV,\ B \in S(V^*)$ by $\langle A, B\rangle \stackrel{\text{def}}{=} (\underline{A}B)(0)$ is symmetric, so

$$[\underline{e^I\phi}(e^{-g^*/2})]\ (0)\ =\ [\underline{e^{-g^*/2}}(e^I\phi)]\ (0).$$

(The symmetry persists for $A \in \widehat{S}V,\ B \in \widehat{S}(V^*)$ as long as the pairings are defined – for instance when the coefficients of A are infinitesimals.)

The exponential of I is the sum over $\mathcal{N} \stackrel{\text{def}}{=} \oplus_{k=0}^{\infty} \mathbb{N}$

$$e^I = \prod_{k=0}^{\infty} e^{I_k/k!} = \prod_{k=0}^{\infty} \sum_{n_k=0}^{\infty} \frac{I_k^{n_k}}{(k!)^{n_k}} = \sum_{\vec{n}\in\mathcal{N}} \prod_{k=0}^{\infty} \frac{I_k^{n_k}}{(k!)^{n_k}}.$$

We draw a picture consisting of two rows, the upper row represents the numerator $\phi \prod_{k=0}^{\infty} I_k^{n_k}$ of the $\vec{n}$-summand of $e^I\phi$ and the lower row represents the numerator of the M-summand g^{*M}. (For a moment we forget the constants $\prod_k \frac{1}{(k!)^{n_k}}$ and $\frac{1}{M!}$.)

The type of $\phi \prod_{k=0}^{\infty} I_k^{n_k}$ is drawn as a flower patch which starts with N points labeled by $1, ..., N$ followed by n_k k-valent flowers for $k = 0, 1, ...$ We put $\phi_i \in V^*$ on the point labeled i and $I_k \in S^k(V^*)$ on the unlabeled k-valent flowers. In the lower row we draw M segments, each labeled by g^*.[15]

The process of differentiating g^{*M} by $\underline{\phi_1} \cdot \cdot \cdot \underline{\phi_1} \prod_{k=0}^{\infty} \underline{I_k}^{n_k}$ and then evaluating at $0 \in V$ produces a sum of terms corresponding to choices of bijections ι of points in the upper row (the N labeled point and all the outer ends of petals) and points in the lower row (the ends of segments) – such bijection is a way for labels on the upper row (a bunch of elements of V^* indexed by the upper points) to differentiate the labels in the low row (a bunch of elements of V, considered as linear functions on V^* and indexed by lower points).

A bijection $p' \stackrel{\iota}{\leftrightarrow} p''$ of upper and lower points can be graphically presented by joining points p' and $p'' = \iota(\psi')$ by a segment. This produces a graph with tails γ whose vertices are the roots of flowers in the upper row and each edge between roots r_1 to r_2 is obtained by patching together five parts: two petals at roots r_1 at r_2 (omitted if the valency of the root is 0), two segments given by the bijection and one segment from the lower row. Since γ has N vertices labeled by $1, ..., N$ we can view γ as an object in $\mathcal{G}_N$.

The derivative $\underline{\alpha} v$ of $v \in V$ by $\alpha \in V^*$ is just the contraction $\langle \alpha, v \rangle$, so the result of the differentiation along the bijection ι is precisely the contraction $w_P^\gamma(I, \phi) \stackrel{\text{def}}{=} \langle P_\gamma, (I, \phi)_\gamma \rangle$.

We have now established that the integral is a sum over isomorphism classes $[\gamma] \in \pi_0(\mathcal{G}_N)$ of graphs $\gamma \in \mathcal{G}_N$)

$$\int_V dx\ e^{-\frac{g}{2}(x)+I(x)}\ \phi_1(x) \cdot \cdot \cdot \phi_N(x) \ = \sum_{[\gamma]\in\pi_0(\mathcal{G}_N)} v_\gamma \cdot w_P^\gamma(I, \phi).$$

This involves some constants v_γ which incorporate the constants $\prod_k \frac{1}{(k!)^{n_k}}$ and $\frac{1}{M!}$ above as well as certain repetitions: different bijections ι may produce isomorphic graphs γ. It remains to prove the following lemma:

[15] The more precise meaning of labeling is that the labels I_k are sums of labels of the form $\alpha_1 \otimes \cdot \cdot \cdot \alpha_k$ with $\alpha_i \in V^*$, and we place each factor α_i at one petal of the flower. Because I_k is a symmetric tensor it does not matter how we order the petals. Similarly g^* is a sum of terms $a_1 \otimes a_2$ with $a_i \in V$ and we place a_i at the i^{th} boundary point of the segment.

3.4.4. *Lemma.* The constant v_γ is the volume $\frac{1}{|\mathrm{Aut}(\gamma)|}$ of the class $[\gamma]$ in $\pi_0(\mathcal{G}_N)$.

It is not difficult to prove this lemma directly, i.e., combinatorially by counting the above repetitions. We will deduce it from the categorified version of the above construction.

3.4.5. *Category theory.* The observation that graphs have automorphisms means that the totality of graphs (of a given kind) is not a set but a category. The categorical language turns out to be illuminating in Feynman expansions (though one could do fine without it).

One may make a case that one psychological difference between mathematicians and Fascist's is that the former are raised in Set Theory and latter in Category Theory. The first question about an object that one asks in Set Theory is *What is it?*. In Category Theory the question is *How does it relate to others*?

3.4.6. *Groupoids (stacks) and the notion of moduli.* The "totality" (or "moduli") of objects of a certain kind $\mathcal{K}$ is often in itself an interesting (important) object of study. This totality is most naively viewed as a set of all $\mathcal{K}$-objects (example: various sets of numbers). However, if there is a notion of isomorphism of objects of the kind $\mathcal{K}$ then objects isomorphic to a given object a are naturally viewed as repetitions of a, so a better notion of totality is the set $\pi_0(\mathcal{K})$ of isomorphism classes $[a]$ of $\mathcal{K}$-objects a. It turns out that by passing from $\mathcal{K}$ to the set $\pi_0(\mathcal{K})$ the information that we keep is too little, i.e., computations do not work well. The efficient strategy is to consider the totality of $\mathcal{K}$-objects as the category whose objects are $\mathcal{K}$-objects and whose morphisms are the isomorphisms of $\mathcal{K}$-objects.

The categories we get in this way are the *groupoid categories* – categories such that all morphisms are isomorphisms ("invertible"). Another name for groupoid categories is or *stacks*.

Example. The moduli of objects of a given category $\mathcal{C}$ is the groupoid category $\mathcal{C}^*$ obtained by throwing away the non-invertible morphisms, so objects are the same as before $Ob(\mathcal{C}^*) = Ob(\mathcal{C})$, and morphisms $Mor(\mathcal{C}^*) = Isom(\mathcal{C})$ are the isomorphisms in $\mathcal{C}$.

Example. For instance, the moduli $\mathcal{FS}$ of finite sets is the groupoid part $\underline{\mathcal{FS}}^*$ of the category $\underline{\mathcal{FS}}$ of finite sets.

3.4.7. *Integrals over stacks.* A functor $F : \mathcal{C} \longrightarrow S$ from a stack $\mathcal{C}$ to a set S is the same as a function $f : \pi_0(\mathcal{C}) \longrightarrow S$. We will also call it a *function from* $\mathcal{C}$ *to* S.

For any stack $\mathcal{C}$ the set $\pi_0(\mathcal{C})$ has a canonical measure, for any object x of $\mathcal{C}$ the measure of the isomorphism class $[x] \in \pi_0(\mathcal{C})$ is $1/|\mathrm{Aut}_C(x)| \geq 0$.

The integral of a $\mathbb{C}$-valued function f on a stack $\mathcal{C}$ is defined as the integral of the corresponding function $f : \pi_0(\mathcal{C}) \to \mathbb{C}$, i.e., the sum over isomorphism classes $[x]\pi_0(\mathcal{C})$

of objects x in C

$$\int_{\mathcal{C}} dx\ f(x) \stackrel{\text{def}}{=} \sum_{[a]\in\pi_0(\mathcal{C})} \frac{1}{|\mathrm{Aut}(a)|} f(a).$$

The categorical nature of the construction is reflected in the *change of variable formula*: if a functor $F : \mathcal{D} \to \mathcal{C}$ is an em equivalence of categories, then $\int_{\mathcal{D}} f \circ F = \int_{\mathcal{C}} f$.

3.4.8. *Monoidal categories (3.8).* A binary operation on a category $\mathcal{C}$ is a functor $\mathcal{C}\times\mathcal{C} \stackrel{m}{\to} \mathcal{C}$. We say that $(\mathcal{C}, m)$ is a *monoidal category* if m is a associative and unital in an appropriate sense. A *tensor category* is a monoidal category such that m is commutative (again, in an appropriate sense).

Just as to any set S we can associate a free group generated by S, to any category $\mathcal{C}$ we can associate a free tensor category $\mathcal{C}^{\otimes}$ generated by $\mathcal{C}$,

Example. The moduli $\mathcal{FS}$ of finite sets is a tensor category with the operation $\sqcup$ of disjoint union. Actually, $(\mathcal{FS}, \sqcup)$ is (equivalent to) the free tensor category $\mathrm{pt}^{\otimes}$ generated by the "point" category pt which has one object and one morphism.

More generally, we will consider various categories of graphs $\mathcal{G}, \mathcal{TG}, \mathsf{Pai}, \mathsf{Par}, ...$ as tensor categories for the operation $\sqcup$ of disjoint union. All these will be considered as tensor categories over $(\mathcal{FS}, \sqcup)$ via projection to prongs. The prefix "c" will mean the submoduli of nonempty connected graphs.

Lemma. For any class of graphs $\mathfrak{G}$ closed under finite disjoint unions, the moduli $(\mathfrak{G}, \sqcup)$ is a free tensor category $c\mathfrak{G}^{\otimes}$ generated by the submoduli $c\mathfrak{G}$ of connected nonempty graphs in $\mathfrak{G}$

In particular, the following are free tensor categories

(1) $(\mathsf{Par}, \sqcup) \cong c\mathsf{Par}^{\otimes} \cong \mathcal{FS}^{\otimes}$ since the connected partitions are the same as finite sets: $c\mathsf{Par} \cong \mathcal{FS}$.
(2) $(\mathsf{Pai}, \sqcup) \cong c\mathsf{Pai}^{\otimes} \cong \mathcal{FS}_2{}^{\otimes}$ since the connected pairings are the same as 2-element sets: $c\mathsf{Pai} \cong \mathcal{FS}_2$.
(3) $(\mathcal{G}, \sqcup) \cong c\mathcal{G}^{\otimes}$ and $(\mathcal{TG}, \sqcup) \cong c\mathcal{FG}^{\otimes}$.

3.4.9. *A combinatorial principle (see 3.8).* It says that the integrals over connected graphs are logarithms of the corresponding integrals over all graphs. The reason is really that for a class of graphs $\mathfrak{G}$ (closed under finite disjoint unions), the moduli $(\mathfrak{G}, \sqcup)$ is a free tensor category $c\mathfrak{G}^{\otimes}$ generated by the submoduli $c\mathfrak{G}$ of connected nonempty graphs in $\mathfrak{G}$ (see lemma). Our principle is then an example of the following property of the construction of a free tensor category $(\mathcal{C}^{\otimes}, \sqcup)$ from a category $\mathcal{C}$.

Lemma. Any function $w : \mathcal{C} \to \mathbb{C}$ on $\mathfrak{G}$ extends to a multiplicative function $(\mathcal{C}^{\otimes}, \sqcup) \xrightarrow{w^{\otimes}} (\mathbb{C}, \cdot)$, and then

$$\int_{\mathcal{C}^{\otimes}} w^{\otimes} \;=\; exp(\int_{\mathcal{C}} w).$$

3.4.10. *Fibered products of tensor categories.* The *fibered product* $\mathcal{A}\times_{\mathcal{C}}\mathcal{B}$ of the diagram of categories $\mathcal{A} \xrightarrow{\alpha} \mathcal{C} \xleftarrow{\beta} \mathcal{B}$ is the category of triples (a, b, ι) where $a \in \mathcal{A},\ b \in \mathcal{B}$ and $\iota : \alpha(a) \cong \beta(b)$ is an isomorphism in $\mathcal{C}$.

The fibered product of a diagram of tensor categories $(\mathcal{A}, \otimes) \xrightarrow{\alpha} (\mathcal{C}, \otimes) \xleftarrow{\beta} (\mathcal{B}, \otimes)$, is the fibered product of categories $\mathcal{A}\times_{\mathcal{C}}\mathcal{B}$, with the operation

$$(a, b; \alpha a \underset{\cong}{\xrightarrow{\iota}} \beta b)\otimes(a', b'; \alpha a' \underset{\cong}{\xrightarrow{\iota'}} \beta b') \overset{\text{def}}{=} \big(a\otimes a', b\otimes b'; \alpha(a\otimes a') \underset{\cong}{\xrightarrow{\iota\otimes\iota'}} \beta(b\otimes b')\big)..$$

Lemma. (a) The moduli $\mathcal{G}$ of graphs is (as a tensor category) the fibered product of partitions and pairings over finite sets

$$(\mathcal{G}, \sqcup) \;\cong\; (\mathsf{Par}, \sqcup)\times_{(\mathcal{FS},\sqcup)}(\mathsf{Pai}, \sqcup) \;\cong\; \mathcal{FS}^{\otimes}\times_{(\mathcal{FS},\sqcup)}{\mathcal{FS}_2}^{\otimes}.$$

(a') Similarly, the moduli $\mathcal{G}_N$ of graphs with N univalent vertices colored by $1, ..., N$, is (as a tensor category) the fibered product of the moduli Par_N of partitions with N univalent vertices colored by $1, ..., N$, and the moduli of pairings:

$$(\mathcal{G}_N, \sqcup) \;\cong\; (\mathsf{Par}_N, \sqcup)\times_{(\mathcal{FS},\sqcup)}(\mathsf{Pai}, \sqcup) \;\cong\; \mathcal{FS}^{\otimes}\times_{(\mathcal{FS},\sqcup)}{\mathcal{FS}_2}^{\otimes}.$$

(b) As a tensor category, the moduli $\mathcal{TG}$ of Feynman graphs (graphs with tails) is a the fibered product of pairings and of marked partitions (the marking is a subset of prongs)[16]) over the the tensor category $(\mathcal{FS}, \sqcup)$:

$$(\mathcal{TG}, \sqcup) \quad\cong\quad (\mathsf{Pai}, \sqcup)\times_{(\mathcal{FS},\sqcup)} (\mathsf{mPar}, \sqcup).$$

Proof. (a) A graph $\Gamma = (P \xleftarrow{\sigma} P \xrightarrow{\pi} V)$ defines a a compatible triple of a pairing $(\mathcal{P} \to \mathcal{P}/\{1, \sigma\}) \in \mathsf{Pai}$, a partition $(P \xrightarrow{\pi} V) \in \mathsf{Par}$, and a set $P \in \mathcal{FS}$. This gives the map $\mathcal{G} \to \mathsf{Pai}\times_{\mathcal{FS}}\mathsf{Par}$.

In the opposite direction, an object of $\mathsf{Pai}\times_{\mathcal{FS}}\mathsf{Par}$ consists of a pairing (P'', Γ), a partition $(P' \xrightarrow{\pi} V)$ and an isomorphism $\iota : P' \xrightarrow{\cong} P''$ in $\mathcal{FS}$. It defines a graph $\Gamma = (P \xleftarrow{\sigma} P \xrightarrow{\pi} V)$ with $P = P'$ and the involution σ characterized by $\{\iota p, \iota\sigma p\}$ for all $p \in P$.

The proof of (b) is the same.

[16]This is the subset of prongs which will be connected to prongs of the pairing. The remaining prongs represent the remaining degrees of freedom.

3.4.11. *Proof of lemma 3.4.4.* We will state the proof for the case when $\phi = 1$, i.e., only the calculation of the free partition function. The general case of expectation integrals is proved the same except that one uses part (a') of the lemma 3.4.10 rather than the part (a).

First, $I = \sum_{i=0}^{\infty} I_k/k!$ can be viewed as $\int_{\mathcal{FS}} I_\bullet$ where $I_\bullet : \mathcal{FS} \to SV^* \supseteq \widehat{S}(V^*)$ sends $A \in \mathcal{FS}$ to $I_\bullet(A) \stackrel{\text{def}}{=} I_{|A|}$. Similarly, $e^{g^*} = \int_{\mathcal{FS}_2} g^*$ where $g^* : \mathcal{FS}_2 \to SV$ is the constant functor $g^*(A) = g^* \in S^2V$, $A \in \mathcal{FS}_2$. Now,

$$\langle e^{g^*/2}, e^I \rangle \;=\; \langle e^{\int_{\mathcal{FS}} {I_\bullet}},\; e^{\int_{\mathcal{FS}_2} \mathsf{g}} \rangle \;=\; \langle \int_{\mathcal{FS}^\otimes} {I_\bullet}^\otimes,\; \int_{\mathcal{FS}_2{}^\otimes} \mathsf{g}^\otimes \rangle$$

$$= \langle \int_{\mathsf{Par}} {I_\bullet}^\otimes,\; \int_{\mathsf{Pai}} \mathsf{g}^\otimes \rangle \;=\; \int_{\mathsf{Par}\ni\alpha} \int_{\mathsf{Pai}\ni\beta} \langle {I_\bullet}^\otimes(\alpha),\; \mathsf{g}^\otimes(\beta) \rangle \;=\; ,$$

The integrand, i.e., the pairing

$$\langle {I_\bullet}^\otimes(\alpha),\; \mathsf{g}^\otimes(\beta) \rangle = \underline{{I_\bullet}^\otimes(\alpha)} \mathsf{g}^\otimes(\beta) \;(0)$$

is a sum of ι-contractions

$$\int_{\iota \in \mathrm{Isom}_{\mathcal{FS}}[\mathrm{prongs}(\alpha), \mathrm{prongs}(\beta)]} \langle {I_\bullet}^\otimes(\alpha), \mathsf{g}^\otimes(\beta) \rangle_\iota$$

over bijections ι of prongs of the partition α and prongs of the pairing β. The triple (α, β, ι) is the same as a graph $\gamma \in \mathcal{G}$, and then the ι-contraction $\langle {I_\bullet}^\otimes(\alpha), \mathsf{g}^\otimes(\beta) \rangle_\iota$ is just the definition $\langle I_\gamma, P_\gamma \rangle_\gamma$ of the amplitude $w^\gamma_{g^*}(I)$. So, we get

$$\int_{\mathcal{G}\ni\gamma} w^\gamma_{g^*}(I).$$

3.5. **Variations of the Feynman expansion.** We will also consider refinements of the Feynman expansion when I is allowed to be a function of $\hbar$ (3.5.4) or when the space of fields is graded (3.9.2). The effect of the additional structure is that summands decompose into finer terms which are indexed by graphs with additional data.

Finally, we consider the effect of an additive shift in the action S in 3.5.5.

3.5.1. *Propagators.* The data $(\mathcal{E}, g, I)$ for an abstract Feynman expansion of correlator integrals consist of a finite dimensional vector space $\mathcal{E}$, a positive metric g on $\mathcal{E}$ and an infinitesimal function I on $\mathcal{E}$. The correlator integral $\langle \phi \rangle_{-\frac{g}{2}+I}$ is then calculated in terms of the dual metric g^* on $\mathcal{E}^*$ as

$$\langle \phi \rangle_{g^*, I} \;\stackrel{\text{def}}{=}\; \int_{\mathcal{G}_N \ni \gamma} w^\gamma_{g^*}(I, \phi).$$

We notice that the construction $\langle\phi\rangle_{g^*,I}$ generalizes from metrics g^* to arbitrary symmetric tensors $\mathsf{P} \in S^2\mathcal{E}$. The expression

$$\langle\phi\rangle_{\mathsf{P},I} \stackrel{\text{def}}{=} \int_{\mathcal{G}_N \ni \gamma} w_{\mathsf{P}}^{\gamma}(I,\phi)$$

still makes sense since one can still define the tensors $\mathsf{P}_\gamma \in SV$ and contract them with tensors $(I,\phi)_\gamma \in S(V^*)$ to define the numbers $w_{\mathsf{P}}^{\gamma}(I,\phi) \stackrel{\text{def}}{=} \langle \mathsf{P}_\gamma, (I,\phi)_\gamma\rangle$.

3.5.2. *The logarithm W of the partition function ("free energy") and the connected Feynman graphs.* Taking logarithm does not seem reasonable for correlators, we restrict ourselves to the partition function

$$Z \;=\; \int_V dx\; e^{-\frac{g}{2}+I} \;=\; \int_{\gamma\in\mathcal{G}} w_P^{\gamma}(I).$$

Then the constant term (i.e., modulo infinitesimals) is 1, so the logarithm is defined.

Theorem. (a) The formal logarithm $W = \ln(Z)$ is given by the same integral but now restricted to the connected nonempty graphs:

$$W \;=\; \int_{\gamma\in c\mathcal{G}} w_P^{\gamma}(I).$$

(b) When free energy is viewed as a function $W(I)$ of the interaction I, its linearization $d_I W$ is a linear functional on $T_I(\mathit{interactions})$ which is the space of (naive) quantum observables. The value of the differential $d_I W$ on an observable O (the directional derivative of W at I in the direction of O) is the (normalized!) expectation

$$(d_I W)\mathsf{O} \;=\; \frac{\langle\mathsf{O}\rangle_{g,I}}{\langle 1\rangle_{g,I}}.$$

Proof. (a) This is a case of lemma 3.4.9 since $\mathcal{G}$ is the free tensor category $c\mathcal{G}^{\otimes}$ generated by category $c\mathcal{G}$ and $w_P^{\gamma'\sqcup\gamma''}(I) = w_P^{\gamma'}(I)\cdot w_P^{\gamma''}(I)$.[17]

(b) $W(I) \;=\; \log[\int_V\; e^{-\frac{g}{2}+I}$, hence

$$(d_I W)(\mathsf{O}) \;=\; \frac{1}{\int_V\; e^{-\frac{g}{2}+I}]}\; \int_V\; e^{-\frac{g}{2}+I}\mathsf{O} = \; \frac{\langle\mathsf{O}\rangle_{g,I}}{\langle 1\rangle_{g,I}}.$$

[17] This last property fails for expectation amplitudes $w_P^{\gamma}(I,\phi)$.

Remarks. (0) Similarly, the correlators $(\mathsf{O}_1, ..., \mathsf{O}_n) \mapsto \langle \mathsf{O}_1, ..., \mathsf{O}_n \rangle_{g,I}$ are the multiderivatives of W at I.

(1) This shows that the free energy construction $I \mapsto W(I)$ is the fundamental object – the true correlators are just its Taylor expansions.[18]

(2) Also, it shows that observables naturally appear as first order deformations of the action!

3.5.3. *Adding the Planck constant $\hbar$: loop expansion of free energy and classical limits.* Here we consider how correlators change when we vary the action according to the *scale* $\hbar > 0$. We replace the action S by $\frac{1}{\hbar}S$, so the expectation integral becomes

$$\langle \phi \rangle_\hbar \stackrel{\text{def}}{=} \int_V dx\ e^{\frac{1}{\hbar}S[v]}\phi = \int_V dv\ e^{\frac{1}{\hbar}(-\frac{g}{2}+I)}\phi.$$

A. *Adding the Planck constant to correlators.* As we will see, if taken literally, this would only be reasonable in finite dimensional Feynman integrals.

Lemma. The $\hbar$-expectation integral of a homogeneous polynomial ϕ is (χ_γ is the Euler characteristic of the graph γ)

$$\langle \phi \rangle_\hbar \ = \ \hbar^{\frac{1}{2}[\deg(\phi)+\dim(V)]} \int_{\mathcal{G}_N \ni \gamma} \hbar^{-\chi_\gamma}\ w_P^\gamma(I, \phi).$$

Proof. In order to reduce the integral to the kind we had before, we change the variable by $x = \hbar^{\frac{1}{2}} \cdot u$, so that $dx = \hbar^{\frac{1}{2}\dim(V)} du$ and

$$\frac{S(x)}{\hbar} \ = \ -\frac{g(\hbar^{\frac{1}{2}}u)}{2\hbar} + \frac{1}{\hbar}\sum_k \frac{I_k(\hbar^{\frac{1}{2}}u)}{k!} \ = \ -\frac{g(u)}{2} + \frac{1}{\hbar}\sum_k \hbar^{\frac{1}{2}k}\frac{I_k(u)}{k!}.$$

Therefore, the $\hbar$-expectation is

$$\langle \phi \rangle_\hbar \ = \ \int_V du\ \hbar^{\frac{1}{2}\dim(V)} \cdot e^{-\frac{g(u)}{2}+\frac{1}{\hbar}I(\hbar^{\frac{1}{2}}u)} \cdot \phi(\hbar^{\frac{1}{2}}u)$$

$$= \ \hbar^{\frac{1}{2}[\dim(V)+\deg(\phi)]} \int_V du\ e^{-\frac{g(u)}{2}+\frac{1}{\hbar}\sum_k \hbar^{\frac{k}{2}}\frac{I_k(u)}{i!}}\phi(u).$$

The second factor is the standard expectation integral except that I_k is replaced by $\hbar^{-1} \cdot \hbar^{\frac{i}{2}}$. This means that for $\gamma \in \mathcal{G}_N$, the amplitude $w_P^\gamma(I, \phi)$ has been multiplied by a certain power of $\hbar$. We count the contributions to this power in the following way: each uncolored vertex contributes -1 (from $I_k \mapsto \frac{1}{\hbar}I_i$ since tensors I_k are placed at vertices) and each prong contributes $\frac{1}{2}$ (from $I_i \mapsto \hbar^{\frac{i}{2}}$ since k is the number of prongs at the vertex at which we placed I_k). The total is the number of edges (each edge contains two prongs) minus the number of vertices, i.e., $-\chi_\gamma$.

[18] Free energy will later generalize to operators W_P on the space of interactions.

B. *Loop expansion of free energy.* While in the expectation integrals the power of $\hbar$ was the negative of the Euler characteristic of a graph, the powers of $\hbar$ in the normalized free energy are governed by the number $b_1(\gamma)$ (the first Betti number) of independent loops in the graph.

In order to have a quantity meaningful in the infinite dimensional setting we normalize the expectations to $\hbar^{-\frac{1}{2}\dim(V)}\langle\phi\rangle_\hbar$. In particular, we normalize the partition function $Z_\hbar$ to

$$\widetilde{Z}_\hbar \stackrel{\text{def}}{=} \hbar^{-\frac{1}{2}\dim(V)}\cdot Z_\hbar = \int_{\mathcal{G}} \hbar^{-\chi_{\frac{1}{2}}}\, w_P^\gamma(I).$$

Corollary. The $\hbar$-expansion of the logarithm of the normalized partition function is governed by the number of loops in a graph

$$\log(\widetilde{Z}_\hbar) = \int_{\gamma\in c\mathcal{G}} \hbar^{-\chi_\gamma}\, w_P^\gamma(I) = \sum_{n\geq 0} \hbar^{n-1} W_{n-loop}$$

$$= \hbar^{-1}W_{Trees} + W_{1-loop\ graphs} + \hbar\cdot W_{2-loop\ graphs} + \hbar^2\cdot W_{3-loop\ graphs} + \cdots;$$

for

$$W_{n-\text{loop graphs}} = \int_{\gamma\in c\mathcal{G}_{n-\text{loop}}} w_P^\gamma(I)\ ;$$

where $c\mathcal{G}_{n-\text{loop}}\subseteq\mathcal{G}$ is the submoduli of connected nonempty graphs γ with $b_1(\gamma)=n$.

Remarks. (0) The connected graphs without loops are called trees. We sometimes call $b_1(\gamma)$ the *genus* g_Γ of the graph γ.

(1) $\log(\widetilde{Z}_\hbar)$ has at most the first order pole at 0 and the residue at $\hbar=0$ is W_0 also called W_{Trees}.

Proof. Since $\widetilde{Z}_\hbar) = \int_{\gamma\in\mathcal{G}} \hbar^{-\chi_\gamma}\, w_P^\gamma(I)$ and $\hbar^{-\chi_\gamma}\, w_P^\gamma(I)$ is multiplicative in $(\mathcal{G},\sqcup)$, we have $\log(\widetilde{Z}_\hbar) = \int_{\gamma\in c\mathcal{G}} \hbar^{-\chi_\gamma}\, w_P^\gamma(I)$.

For connected graphs the Euler characteristic simplifies since $\chi_\gamma = b_0 - b_1 = 1 - b_1$, hence $-\chi_\gamma = b_1 - 1$.

Theorem. All coefficients in the loop expansion are *classical quantities*, i.e., they have meaning in classical physics (see **??**).

3.5.4. *Allowing Planck constant $\hbar$ in the interaction term I.* We will quantize classical actions S to effective (scaled) quantum actions which are families of actions $S[L]$ valid at various scales L. The free part of the action will not be affected: $S[L] = -\frac{g}{2} + I[L]$, but the interaction terms of quantum actions $S[L]$ will necessarily depend on the Planck constant $\hbar$.

When interaction depends on $\hbar$,

$$I = \ sum\ I_k/k! \quad \text{and} \quad I_k = \sum\ I_{ik}\hbar^i,$$

the Feynman machinery gets refined. The product $e^{I/\hbar} = \prod e^{I_k/k!}$ which gave terms $\prod_k\ I_k^{\sigma_k}$ now becomes $\prod e^{I_{ik}\hbar^i/k!}$ which gives terms $\prod_{i,k}\ (I_{ik}\hbar^i)^{\sigma_{ik}}$.

This means that on an k-valent flower we do not put the whole I_k but only one of its summands $I_{ik}\hbar^i$. The effect is that we need to color the root v of a flower by some $g_v \in \mathbb{N}$ (called the "*genus*" of the vertex), then we put $I_{ik}\hbar^i$ on the flower of valency k and genus i.

So, the corresponding class of graphs $\mathcal{G}_N^\bullet$ are pairs $\Gamma = (\gamma, g)$ where $\gamma \in \mathcal{G}_N$ and the genus function g maps the set V_γ of uncolored vertices of γ to $\mathbb{N}$. We call them *graded graphs* (graded by genus).

Now, the refined Feynman expansion takes form

$$\langle\phi\rangle_{P,I} \ = \ \int_{\Gamma\in\mathcal{G}_N^\bullet} \hbar^{-\chi_\Gamma+\sum_v\ g_v}\ w_P^\Gamma(I,\phi)$$

where amplitudes are defined in the same way. The logarithm of the partition function is then

$$\log[\widetilde{Z}_\hbar(P,I)] \ = \ \int_{\Gamma\in\ {}_c\mathcal{G}_N^\bullet} \hbar^{g_\Gamma}\ w_P^\Gamma(I,\phi);$$

where we only use the connected nonempty graphs' and the *genus* of a graph is defined as

$$g_\Gamma \ \stackrel{\text{def}}{=}\ b_1(\Gamma) + \sum_{v\in V_\Gamma}\ g_v.$$

3.5.5. *Correlators as functions on fields ("integrals with additive shift").* Here we notice that Feynman's correlator integrals are a recording of a flow on the space of fields generated by the constant coefficient differential operator $\frac{1}{2}\underline{g^*}$.

(A) *Extension of correlators to functions on infinitesimal fields (by additive shifts).* Here we promote correlators from numbers to functions defined on infinitesimal elements a of V, i.e., points of the formal neighborhood $\widehat{V}$ of 0 in V. This is done by shifting the interaction function I and the observable ϕ by a :

$$\langle\phi\rangle_{g,I}(a) \stackrel{\text{def}}{=} \ \int_V\ dx\ e^{-\frac{g(x)}{2}+I(a+x)}\phi(x+a).$$

This is a deformation of the usual correlator $\langle\phi\rangle_{g,I}$ which is now written as $\langle\phi\rangle_{g,I}(0)$. In particular, we have promoted the partition function $Z_{g,I}$ to a function of $a \in \widehat{V}$ by

$$Z_{g,I}(a) \stackrel{\text{def}}{=} \ \int_V\ dx\ e^{-\frac{g(x)}{2}+I(a+x)}.$$

Remark. Alternatively, this means that the Gaussian probability distribution

$$e^{-\frac{g(x)}{2}+I(a+x)}$$

has been shifted by a. (This is also the most natural point of view with respect to signs of translations.) □

(B) *Extension of graph-wights to polynomial functions on fields (by allowing tails).* Let $\mathcal{TG}$ be the class of *graphs with tails*. We extend the graph weight construction $\gamma \mapsto w^{\gamma}_{g^*}(I;\phi)$ from graphs to graphs with tails. For $\gamma \in \mathcal{TG}$ we define the amplitude $w^{\gamma}_{g^*}(I;\phi)$ as a homogeneous polynomial function on fields (of degree t_γ), given by the γ-contraction

$$w^{\gamma}_{g^*}(I;\phi) \stackrel{\text{def}}{=} \langle (g^*)_\gamma, (I,\phi)_\gamma \rangle_\gamma$$

of the tensor $(I,\phi)_\gamma \in S(V^*)$ by the tensor $(g^*)_\gamma \in SV$.

Remark. The generalized notion of weights applies to a larger class of objects and has a more sophisticated values (functions rather than just constants). On the other hand, the extension in **(A)** was only in the direction of more sophisticated values (again, functions rather than just constants).

(C) $\underline{g^*}$*-flow on fields.* We will loosely talk of infinite order differential operators

$$\Phi_t \stackrel{\text{def}}{=} e^{\frac{-1}{2t}\underline{g^*}}$$

as "stochastic flow by $\frac{1}{t}\underline{g^*}$" or the "time t $\underline{g^*}$-flow" on the space of fields.

Remark. For $t > 0$ this is justified (at least under some positivity and in the finite dimensional case) by the heat kernel theory, the heat kernel operator is then interpreted as the pull-back of fields under the flow

$$K_t x \;=\; \Phi_t^* x, \qquad x \in \mathcal{E}.$$

Lemma. The correlator function $\langle\phi\rangle_{g,I}(a)$ is well defined for infinitesimal elements a of V and can be represented as the pull-back under the $-\underline{g^*}$-flow:

$$\langle\phi\rangle_{g,I} \;=\; e^{\frac{1}{2}\underline{g^*}}(e^I\phi).$$

Proof. Denote the translation by a as $(T_a f)(x) \stackrel{\text{def}}{=} f(x-a)$.

Since a is infinitesimal, $(T_{-a}I)(x) = I(a+x)$ is a well defined formal function on fields. The Feynman expansion for shifted quantities $T_{-a}I$ and $T_{-a}I\phi$

$$\int_V dx\; e^{-\frac{g(x)}{2}+I(x+a)}\phi(x+a) \;=\; \int_V dx\; e^{-\frac{g}{2}+I_a}\phi_a \;=\; \int_{\gamma\in\mathcal{G}} w^{\gamma}_{g^*}(I_a;\phi_a)$$

can be computed (see the proof 3.4.3 of Feynman expansion) as $\underline{e^{g^*/2}}(e^{I_a}\phi_a)\,(0)$. However, since the differential operator g^* is translation invariant, this is just the evaluation of the derivative $\underline{e^{g^*/2}}(e^I\phi)$ at a.

(D) *The graph expansion.*

Theorem. The correlator functions have Feynman expansion over graphs with tails

$$\langle\phi\rangle_{g,I}(a) \;=\; \int_{\gamma\in\mathcal{TG}} w^{\gamma}_{g^*}(I;\phi)(a).$$

Proof. We work on the expression from the lemma

$$\langle\phi\rangle_{g,I} \;=\; e^{\frac{1}{2}\underline{g^*}}(e^I\phi).$$

In 3.4.3 we saw that the computation of $\underline{e^{g^*/2}}e^I$ breaks into differentiation of products $\underline{\phi_1}\cdots\underline{\phi_1}\prod_{k=0}^{\infty}\underline{I_k}^{n_k}$ by powers g^{*M}. This time we evaluate the derivatives at a rather than at $0\in V$, the effect is that the summands we get are indexed by all injections ι of the *lower* row (representing g^{*M}) into the upper row, rather than only the bijections ι.

Therefore, some of the prongs in the upper row may remain free and here we will plug in a. The graphs that we get in this way will have "tails" and will therefore represent functions on $\mathcal{E}$, which we then evaluate at a.

3.6. **Appendix A. Feynman graphs.** We will consider graphs as combinatorial objects or equivalently, as geometric objects. We pass between the two points of view whenever convenient.

3.6.1. *Graphs with tails.* A graph is usually said to consist of vertices connected by edges. For the correct notion of automorphisms of graphs we will take the point of view that a graph consists of vertices, edges and *prongs* (the half-edges).[19]

Beyond the class $\mathcal{G}$ of graphs we will need a larger class $\mathcal{TG}$ of *Feynman graphs* or *graphs with tails*, which is closed under the operation of cutting edges at the midpoint:

A Feynman graph ("F-graph") is a diagram of finite sets

$$\gamma \;=\; (P_\gamma \xleftarrow{\sigma} P_\gamma \xrightarrow{\pi} V_\gamma)$$

where σ is an involution. We say that V_γ is the set of vertices and P_γ is the set of prongs. More terminology:

- Prongs partition into the set of *tails* or *external* prongs $T_\gamma = (P_\gamma)^\sigma$ and the set of *internal* prongs $P_\gamma - T_\gamma$.
- The set of *edges* is the set of free orbits of the involution $E_\gamma \stackrel{\text{def}}{=} (P_\gamma - T_\gamma)/\{1,\sigma\}$.
- For a vertex $v\in V_\gamma$, denote by $P_v = P_{\gamma,v} \stackrel{\text{def}}{=} \pi^{-1}v$ the prongs rooted at v. Then $p_v = |P_v|$ is the *valency* of v.
- $t_\gamma = |T_\gamma|$ is called the *external valency* of Γ.

[19] The non-standard addition of prongs makes difference in the case of loops, a loop now has a nontrivial involution that switches its ends.

Let $\mathcal{TG}$ be the moduli of F-graphs. Adding "c" to the notation (as in $c\mathcal{FG}$), will always mean that we consider the submoduli of *connected nonempty* objects.

Examples. (0) The ordinary graphs appear as the "no tails" subclass $\mathcal{G}$ of $\mathcal{TG}$. This means that one adds the requirement that the involution ι has no fixed points.

(1) We will also use the class $\mathcal{TG}_N$ of graphs with tails endowed with a coloring of N tails by $1, .., N$; and its subclass $\mathcal{G}_N$ in which the only tails are the N colored tails.

Remark. (*Alternative terminology.*) Notice that in our terminology, tails do not end in vertices. However there is another terminology where graph $\Gamma = (P_\Gamma \xleftarrow{\sigma} P_\Gamma \xrightarrow{\pi} V_\Gamma)$ is viewed as data $\underline{\Gamma}$ consisting of (i) vertices $\underline{V_\Gamma} = V_\Gamma \sqcup T_\Gamma$ called *inner* and *outer* vertices, (ii) prongs $\underline{P_\Gamma} = P_\Gamma$ and (iii) edges $\underline{E_\Gamma} = E_\Gamma \sqcup T_\Gamma$ called *inner* and *outer* edges.[20]

3.6.2. *The geometric realization of graphs.* It is defined by

$$|\Gamma| \stackrel{\text{def}}{=} (P_\Gamma \times [0, \frac{1}{2}] \sqcup V_\Gamma) \sim$$

where $\sim$ involves two kinds of identifications, for $h \in P_\Gamma$

- (i) $(h, 0) \sim \pi(h) \in V_\Gamma$,
- (ii) $(h, \frac{1}{2}) \sim (\sigma h, \frac{1}{2})$.

The topological invariants of a graph Γ are defined as topological invariants of $|\Gamma|$.

Lemma. $H_1(|\Gamma|, \mathbb{Z})$ is torsion free.

Proof. One can pass from a graph Γ to a homotopic one by contracting one tail or a edge which is not a loop.[21] Since contracting an edge decreases the number of vertices by one, each connected component Γ is homotopic to a graph with one vertex and no prongs, i.e., a bouquet of circles.

Remark. In physics, $H_1(\Gamma, \mathbb{R})$ is regarded as the moduli of degrees of freedom ("extra variables") which are not determined by applying conservation laws at vertices. These cause appearance of divergent integrals. The case of *forests* ("graphs without loops") is characterized by absence of these extra variables.

[20] I understand that this terminology is used in physics with a different intuition.

[21] But it is not OK to contract several non-loop edges at the same time because contracting the first edge may make the next edge into a loop.

3.6.3. *Partitions (flower patches).* The *graphs without edges* (i.e., all prongs are tails) are the same as maps $P \xrightarrow{\pi} V$ of finite sets, i.e., the same as partitions of P according to V, $P \sqcup_{v\in V} \ \pi^{-1}v$.

Their geometric realizations are called *flower patches* – the connected components are indexed by vertices $v \in V$ and the component at v is the *flower* with the *root* v and the set of *petals* (prongs) $\pi^{-1}v$. The corresponding submoduli of $\mathcal{TG}$ is denoted Par or FP.

Notice that $\pi^{-1}v$ can be empty, then the flower is just one vertex, the root, without petals.

Cutting all edges in a geometric graph at midpoints gives the *flower patch* map of moduli

$$\mathcal{TG} \xrightarrow{\mathsf{fp}} \mathsf{Pai},\ \Gamma \mapsto \mathsf{fp}_\Gamma.$$

Another interesting functor is the *action* map

$$\mathcal{TG} \xrightarrow{a} \mathsf{FP},\ \Gamma \mapsto a_\Gamma = \overline{\Gamma},$$

where $\overline{\Gamma}$ is obtained from Γ by contracting all edges (this causes some vertices to be identified).

The moduli of flowers, i.e., the moduli $c\mathsf{Par}$ of connected (nonempty) flower patches, is equivalent to the moduli of finite sets via $\mathcal{FS} \xrightarrow{\cong} c\mathsf{Par}$, $S \mapsto \star_S$, where $\star_S$ has one vertex, and its set of prongs is S. Flower patches are the same as *partitions*, i.e., maps of finite sets $P \xrightarrow{\pi} V$ (viewed as a partition of P by V, with some strata possibly empty).

The set of *numerical partitions* $\mathcal{N} = \oplus_{n\in\mathbb{N}}\ \mathbb{N}$ consists of sequences $(\lambda_0, \lambda_1, ...) \in \mathbb{N}^{\mathbb{N}}$ with compact supports. The *order* of a partition $(P \xrightarrow{p} V) \in \mathsf{Par}$ is the numerical partition $\vec{n}_{P\xrightarrow{p}V} \stackrel{\text{def}}{=} (n_0, n_1, ...)$ where n_k is the number of $v \in V$ with $|p^{-1}v| = k$. This gives the *order* function $\vec{n} :\ \mathcal{P} \to \mathcal{N}$ on the moduli of partitions.

3.6.4. *Pairings.* By a *pairing* on a finite set A we mean a partition $A \xrightarrow{p} V$ of A into 2-element subsets. So, their moduli Pai is a submoduli of partitions: $\mathsf{Pai} \subseteq \mathsf{Par} \subseteq \mathcal{TG}$ and the geometric realizations are binary flower patches, i.e., such that each flower has precisely two petals.

However, in Feynman calculus it is traditional to use a different interpretation of pairings as graphs, i.e., a different embedding $\mathsf{Pai} \hookrightarrow \mathcal{G} \subseteq \mathcal{TG}$. So, these graphs are simply disjoint unions segments, i.e., graphs without tails such that each vertex v lies in precisely one edge and this edge is not a loop. The relation between the two is that to a pairing $A \xrightarrow{p} V$ one attaches the graph Γ with the set of vertices A and edges given by V.

3.7. **Appendix B. Stacks.**

3.7.1. This is an introduction to the language of stacks.

3.7.2. *Categories.* Notice that categories form a "category of all categories" $\mathcal{C}at$ with morphisms $\mathrm{Hom}_{\mathcal{C}at}(\mathcal{A},\mathcal{B})$ given by functors: $Fun(\mathcal{A},\mathcal{B})$.

3.7.3. *Stacks.* This is a certain class of spaces which generalize sets. The need for stacks arises formally by a need of for the *correct* quotient X/G when a group G acts on a set X. A more sophisticated origin of stacks is the *moduli problem.* **CHECK** The same.

The formal definition of stacks will be an algebraic one. A stack $\mathcal{S}$ is a *groupoid category*, i.e., a category such that all morphisms are invertible.[22] The geometric view of $\mathcal{S}$ is to view it as the quotient space $Ob(\mathcal{S})/Mor(\mathcal{S})$, i.e., the set of objects of $\mathcal{S}$ taken "modulo all identifications given by morphisms in $\mathcal{S}$".

Example. Any group G defines a stack $B(G)$ called the classifying space of G. This is the category with one object pt ("a point") and $\mathrm{Hom}(\mathrm{pt},\mathrm{pt}) = G$. As a space this is the quotient $B(G) = \mathrm{pt}/G$ of a point by the (trivial) action of the group G.

Any stack is equivalent to a disjoint unions of classifying spaces $\sqcup_{i\in I}\, B(G_i)$. For all stacks that we will consider the groups G_i are finite.

Notation. We denote the set of orbits of a group G on a set X by $X//G$ while X/G means the stack quotient of X by G, i.e., $Ob(X/G) = X$ and $\mathrm{Hom}_{X/G}(x,y) = \{g \in G;\ gx = y\}$.

3.7.4. *Moduli stacks.* The expression "*moduli stack of objects in a category* C" literally means the corresponding *groupoid subcategory* $C^*\subseteq C$ which has the same objects as C, but morphisms in C^* are now only the isomorphisms in C.

This is refinement of the notion of the *set* $\pi_0(C)$ *of isomorphism classes* of objects in C. This notion usually does not contain enough information because (i) we are forgetting the choices of isomorphisms of objects, (ii) different objects may have different amounts of automorphisms.

3.7.5. *Integrals.* A functor $F : \mathcal{C} \to S$ from a stack $\mathcal{C}$ to a set S is the same as a function $f : \pi_0(\mathcal{C}) \to S$. We will also call it a *function from* $\mathcal{C}$ *to* S.

For any stack $\mathcal{C}$ the set $\pi_0(\mathcal{C})$ has a canonical measure, For any object x of $\mathcal{C}$ the measure of the isomorphism class $[x] \in \pi_0(\mathcal{C})$ is $1/|\mathrm{Aut}_C(x)|$.

Let $\Bbbk$ be a commutative topological ring which contains $\mathbb{Q}$. Then for a $\Bbbk$-valued function f on a stack $\mathcal{C}$, i.e., a functor $f : \mathcal{C} \to \Bbbk$, the integral of f over $\mathcal{C}$ is defined as the integral of the corresponding function $f : \pi_0(\mathcal{C}) \to \Bbbk$, this is the sum over isomorphism classes $[x]\pi_0(\mathcal{C})$ of objects x in C

$$\int_{\mathcal{C}} dx\ f(x) \stackrel{\mathrm{def}}{=} \sum_{[a]\in\pi_0(\mathcal{C})} \frac{1}{|\mathrm{Aut}(a)|} f(a).$$

[22] More precisely this could be called "stacks of sets". There are many other classes of stacks which will not be needed here.

In particular, the size $|\mathcal{C}|$ of a stack $\mathcal{C}$ is defined as

$$|\mathcal{C}| \stackrel{\text{def}}{=} \int_{\mathcal{C}} 1.$$

Remark. When the sum is infinite this definition involves the question of convergence.[23]

3.7.6. *Finite sets.* Let $\mathcal{FS}$ be the moduli of finite sets, i.e., the groupoid category where objects are finite sets and morphisms are bijections. Its connected components $\mathcal{FS}_n$, are the moduli of sets of order n ($n \geq 0$).

The automorphism group of a set A is denoted S_A, in particular $S_n = S_{\langle n \rangle}$ for $\langle n \rangle = \{1, ..., n\}$ is the usual permutation group.

The prefix $\mathcal{O}$, for instance in $\mathcal{OFS}$, will mean "ordered". So, $\mathcal{OFS}$ is the stack of finite ordered sets and $|\mathcal{OFS}_n| = 1$.

In terms of the usual category FS of finite sets (objects are finite sets and morphisms are all functions between sets), $\mathcal{FS}$ is the corresponding groupoid category FS^*.

Lemma. [Baez-Dolan] $|\mathcal{FS}| = e$.

Proof. The category $\mathcal{FS}_n$ is equivalent to the classifying space pt/S_n.

3.7.7. *Global sections of functors on stacks.* Let $\mathcal{S}$ be a stack. For any category $\mathcal{V}$ we consider the category $Fun(\mathcal{S}, \mathcal{V})$ of functors $\mathcal{X} : \mathcal{S} \to \mathcal{V}$ with values in $\mathcal{V}$. When we view $\mathcal{S}$ as a space then the functors in $Fun(\mathcal{S}, \mathcal{V})$ are the same as sheaves on the space $\mathcal{S}$ with values in the coefficient category $\mathcal{V}$.[24]

In the local setting, i.e., when $\mathcal{S} = \mathrm{pt}/G$, a functor $\mathcal{X} : \mathrm{pt}/G \to \mathcal{V}$ is the same as an object $c \in Ob(\mathcal{V})$ (here $c = \mathcal{X}(\mathrm{pt})$), and an action a of G on c (this is the same as a map $a : G = \mathrm{End}_{\mathrm{pt}/G}(\mathrm{pt}) \to \mathrm{End}_C C(c)$ which preserves compositions). When $\mathcal{S} \cong \sqcup_{i \in I} \mathcal{S}_i$ with $\mathcal{S}_I \cong \mathrm{pt}/G_i$ then $Fun(\mathcal{X}, \mathcal{V}) \cong \Pi_{i \in I} Fun(\mathcal{X}_i, \mathcal{V})$.

There are *two* notions $\Gamma(\mathcal{S}, F)$, $\gamma(\mathcal{S}, \mathcal{F})$ of *global sections* of a functor $F : \mathcal{S} \to \mathcal{V}$.

When $\mathcal{S} \cong \{a\}/G$ for a point $\{a\}$ then a functor $F : \mathcal{S} \to \mathcal{V}$ is given by an object $F(a) \in \mathcal{V}$ and an action of G on $F(a)$. Then

$$\Gamma(\frac{\mathrm{pt}}{G}, F) \stackrel{\text{def}}{=} F(a)^G \quad \text{and} \quad \gamma(\frac{\mathrm{pt}}{G}, F) \stackrel{\text{def}}{=} F(a)_G$$

[23] Sometimes we resolve it formally by modifying the natural on $\mathcal{C}$ by a function $g : \mathcal{C} \to \widetilde{\Bbbk}$. For instance, we may take $\widetilde{\Bbbk} = \Bbbk[[g_\alpha,\ \alpha \in \pi_0(\mathcal{C})]]$ if we want to keep all contributions separate.

[24] For the discrete Grothendieck topology on the category $\mathcal{S}$.

are the G-invariants in $F(a)$ and the G-coinvariants for $F(a)$.[25] In general, $\mathcal{S}$ is a disjoint union $\sqcup_{i\in I}\ \mathcal{S}_i$ with $\mathcal{S}_i \cong \{a_i\}/G_i$ and then $\Gamma(\sqcup_{i\in I}\ \mathcal{S}_i, F) = \prod_{i\in I}\ \Gamma(\mathcal{S}_i, F|_{\mathcal{S}})$ and $\gamma(\sqcup_{i\in I}\ \mathcal{S}_i, F) = \sqcup_{i\in I}\ \gamma(\mathcal{S}_i, F|_{\mathcal{S}})$.

3.7.8. *Functoriality of functors on stacks (with respect to change of stacks).* For a map of stacks $\pi : \mathcal{S} \to \mathcal{T}$ there are two direct image constructions $Fun(\mathcal{S},\mathcal{V}) \xrightarrow{\pi_!,\pi_*} Fun(\mathcal{T},\mathcal{V})$ and one inverse image construction $Fun(\mathcal{T},\mathcal{V}) \xrightarrow{\pi^!=\pi^*} Fun(\mathcal{S},\mathcal{V})$.

The inverse image means that $G : \mathcal{T} \to \mathcal{V}$ gives $G\circ\pi : \mathcal{S} \to \mathcal{V}$ which is denoted in two ways as $\pi^! G = \pi^* G$. Locally, π is of the form $\mathrm{pt}/G \to \mathrm{pt}/G'$ for a map of groups $f : G \to G'$ and then the pull-back functor $\pi^! = \pi^*$ just means that we regard a G'-equivariant object $G(\mathrm{pt})$ as a G-equivariant object which we denote $\pi^* G(\pi)$.

The direct image functors are more interesting. In the case of a map to a point $\pi : \mathcal{S} \to \mathcal{T} = \mathrm{pt}$ then $Fun(\mathrm{pt},\mathcal{V}) = \mathcal{V}$ so $\pi_!, \pi_* :\ Fun(\mathcal{S},\mathcal{V}) \to\ \mathcal{V}$. These are defined as the two functors of global sections; $\pi_* F \stackrel{\mathrm{def}}{=}\ \Gamma(\mathcal{S},\mathbb{F})$ and $\pi_! F \stackrel{\mathrm{def}}{=}\ \gamma(\mathcal{S},\mathbb{F})$.

We require properties of operations $\pi_!, \pi_*$

$$(\pi_* \sqcup_{i\in\pi_0\mathcal{C}}\ W|_i)(y) = \prod_{i\in\pi_0\mathcal{C}} (\pi_* W|_i)(y) \quad \text{and} \quad (\pi_!\ \sqcup_{i\in\pi_0\mathcal{C}}\ W|_i)(y) = \sqcup_{i\in\pi_0\mathcal{C}}\ (\pi_* W|_i)(y);$$

and then it suffices to describe these direct image functors in the local situation. Any map $\pi : \mathcal{S} \to \mathcal{T}$ is locally of the form $\mathrm{pt}/G \to \mathrm{pt}/G'$ for a map of groups $f : G \to G'$. The most familiar case is when $\mathcal{V}$ is the category $Vec_{\mathbb{C}}$ of vector spaces. Then $Fun(\mathrm{pt}/G, Vec_{\mathbb{C}})$ is the category of representations of groups and the functoriality is the usual functoriality of representations of groups. The general case of arbitrary $\mathcal{V}$ just imitates these familiar constructions.

Functoriality is compatible with compositions, i.e., for $\mathcal{S} \xrightarrow{\pi} \mathcal{T} \xrightarrow{\sigma} \mathcal{U}$ we have $\sigma_!\circ\pi_! \cong (\sigma\circ\pi)_!$ and $\sigma_*\circ\pi_* \cong (\sigma\circ\pi)_*$. So, by factoring f through its image $f = (G \overset{f'}{\twoheadrightarrow} Im(f) \overset{f''}{\subseteq} G''$ it suffices to describe functors $\pi_!, \pi_*$ in the cases when f is either a quotient map or an inclusion.

In the quotient case $G' \cong G/H$ then for $\pi : \mathrm{pt}/G \to \mathrm{pt}/G'$ the functors $\pi_!, \pi_* : Fun(\mathrm{pt}/G,\mathcal{V}) \to Fun(\mathrm{pt}/G')$ are the H-invariants functor and the H-coinvariants functor, i.e.,

$$(\pi_* F)(\mathrm{pt}) \ =\ F(\mathrm{pt})^H \quad \text{and} \quad (\pi_! F)(\mathrm{pt}) \ =\ F(\mathrm{pt})_H.$$

In the inclusion case have a subgroup $G \overset{f}{\subseteq} G'$ and the functors $\pi_!, \pi_* : Fun(\mathrm{pt}/G, \mathrm{pt}/G')$ are the (co)induction functors,

$$(\pi_* F)(\mathrm{pt}) \ =\ Coind_G^{G'}[F(\mathrm{pt})] \ =\ [\Pi_{y\in G'}\ F(\mathrm{pt})]^G$$

[25] Provided that these (co)invariants exist in the category $\mathcal{V}$. Otherwise, $F(a)^G$ and $F(a)_G$ are only defined as certain *limits*, i.e., as pro-objects and ind-objects of $\mathbb{V}$.

and

$$(\pi_! F)(\mathrm{pt}) \;=\; Ind_G^{G'}\, F(\mathrm{pt}) \;=\; [\sqcup_{y\in G'}\, F(\mathrm{pt})]_G.$$

3.7.9. *(Co)sections in a monoidal category.* This is based on the "(co)section" construction in monoidal categories. The unit object $1_{\mathcal{V}}$ in a monoidal category $\mathcal{V}$ (co)represents the covariant functor $\Gamma_{\mathcal{V}}(v) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(1_{\mathcal{V}}, v)$ ("sections") and the contravariant functor $\gamma_{\mathcal{V}}(v) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(v, 1_{\mathcal{V}})$ ("cosections"). When $\mathcal{V}$ is a $\Bbbk$-linear category, then the values are in the category $\mathfrak{m}(\Bbbk)$ of $\Bbbk$-modules and we can denote these functors by

$$\Gamma_{\mathcal{V}/\Bbbk} :\; \mathcal{V} \to \mathfrak{m}(\Bbbk) \quad \text{and} \quad \gamma_{\mathcal{V}/\Bbbk} :\; \mathcal{V}^o \to \mathfrak{m}(\Bbbk).$$

Lemma. (a) The following diagram is canonically commutative

$$\begin{array}{ccc} Fun(\mathcal{C},\mathcal{V}) & \xrightarrow{\Gamma(\mathcal{C},-)} & \mathcal{V} \\ {\scriptstyle \Gamma_{\mathcal{V}/\Bbbk}}\big\downarrow & & {\scriptstyle \Gamma_{\mathcal{V}/\Bbbk}}\big\downarrow \\ Fun(\mathcal{C},\mathfrak{m}(\Bbbk)) & \xrightarrow{\Gamma(\mathcal{C},-)} & \mathfrak{m}(\Bbbk). \end{array}$$

(b) The functors $\Gamma_{\mathcal{V}/\Bbbk}$ comes with the canonical morphisms of bifunctors

$$\Gamma_{\mathcal{V}/\Bbbk}(A)\otimes_{\Bbbk}\Gamma_{\mathcal{V}/\Bbbk}(B) \to \Gamma_{\mathcal{V}/\Bbbk}(A\otimes_{\mathcal{V}} B)$$

where $A, B \in \mathcal{C}$ and $X, Y : \mathcal{C} \to \mathcal{V}$.

Proof. (a) When $\mathcal{C} = \mathrm{pt}/G$ then for $W \in Fun(\mathcal{C},\mathcal{V})$,

$$\Gamma_{\mathcal{V}/\Bbbk}\Gamma(\mathcal{C}, W) = \mathrm{Hom}_{\mathcal{V}}[1_{\mathcal{V}}, W(\mathrm{pt})^G] \quad \text{and} \quad \Gamma(\mathcal{C}, \Gamma_{\mathcal{V}/\Bbbk} W) = \mathrm{Hom}_{\mathcal{V}}[1_{\mathcal{V}}, W(\mathrm{pt})]^G.$$

These coincide since the action of G on $1_{\mathcal{V}}$ is trivial.

(b) The map is

$$\mathrm{Hom}_{\mathcal{V}}(1_{\mathcal{V}}, A)\otimes_{\Bbbk}\mathrm{Hom}_{\mathcal{V}}(1_{\mathcal{V}}, B) \;\to\; \mathrm{Hom}_{\mathcal{V}}(1_{\mathcal{V}}\otimes_{\mathcal{V}} 1_{\mathcal{V}}, A\otimes_{\mathcal{V}} B) = \mathrm{Hom}_{\mathcal{V}}(1_{\mathcal{V}}, A\otimes_{\mathcal{V}} B).$$

Remark. We can introduce the notion of $\Bbbk$-*sections of functors* $\mathcal{C} \xrightarrow{F} (\mathcal{V}, \otimes)$ *on a stack* $\mathcal{C}$, which is just the composition

$$\Gamma(\mathcal{C}/\Bbbk, -) :\; Fun(\mathcal{C},\mathcal{V}) \;\to\; \mathfrak{m}(\Bbbk), \quad \Gamma(\mathcal{C}/\Bbbk, -) \;\stackrel{\mathrm{def}}{=}\; \big[Fun(\mathcal{C},\mathcal{V}) \;\xrightarrow{\Gamma(\mathcal{C},-)}\; \mathcal{V} \xrightarrow{\Gamma_{\mathcal{V}/\Bbbk}} \mathfrak{m}(\Bbbk)\big].$$

Then the maps

$$\Gamma(\mathcal{C}, X)\otimes_{\mathcal{V}}\Gamma(\mathcal{C}, Y) \;\to \Gamma(\mathcal{C}, X\otimes_{\mathcal{V}} Y)$$

give canonical morphisms $\Gamma(\mathcal{C}/\Bbbk, X)\otimes_{\Bbbk}\Gamma(\mathcal{C}/\Bbbk, Y) \to \Gamma_{\mathcal{V}/\Bbbk}(X\otimes_{\mathcal{V}} Y)$. Locally, $\mathcal{C} = \mathrm{pt}/G$, X, Y are G-equivariant objects $M, N \in \mathcal{V}$ and then the map is $X^G\otimes_{\mathcal{V}} Y^G \to (X\otimes_{\mathcal{V}} Y)^G$.

Example. When $(\mathcal{V}, \otimes) = (\mathfrak{m}(\Bbbk), \otimes_{\Bbbk})$ then $1_{\mathcal{V}} = \Bbbk$, the objects $M \in \mathcal{V}$ are $\Bbbk$-modules and $\Gamma_{\mathcal{V}/\Bbbk} = \mathrm{Hom}_{\mathfrak{m}(\Bbbk)}(\Bbbk, -)$ is the identity functor on $\mathcal{V}$, so, sections of M are just elements of M. On the other hand, $\gamma_{\mathcal{V}/\Bbbk}$ is the dualization operation, i.e., the cosections of M are the linear functionals on M. In this case, $\Gamma(\mathcal{C}/\Bbbk, -) = \Gamma(\mathcal{C}, -)$.

3.7.10. *"Localization on stacks".* This is just a terminology. By "localization $\mathcal{X}$ of an object X over a space $\mathcal{S}$", one means that $\mathcal{X}$ is an object spread over $\mathcal{S}$ in some sense, and that X is a "totality of $\mathcal{X}$" (in some sense).

Here the space $\mathcal{S}$ will be a stack and $\mathcal{X}$ will then be a functor $\mathcal{X} : \mathcal{S} \to \mathcal{T}$ into some category $\mathcal{T}$. Its "totality" X is then interpreted as an integral $\int_{\mathcal{S}} \mathcal{X}$ of $\mathcal{X}$ or as global sections of certain type of $\mathcal{X}$ over $\mathcal{S}$.

3.8. **Appendix C. Free tensor categories as exponentiation.** Here we state the combinatorial principle that the integrals over connected graphs are logarithms of the corresponding integrals over all graphs. The reason is that the moduli $(\mathfrak{G}, \sqcup)$ of a class of graphs (we need $\mathfrak{G}$ to be closed under finite disjoint unions) is a free tensor category $c\mathfrak{G}^{\otimes}$ generated by the submoduli $c\mathfrak{G}$ of connected nonempty graphs in $\mathfrak{G}$ (see **??**).

The general statement is now that the construction of a free tensor category $(\mathcal{C}^{\otimes}, \sqcup)$ from a category $\mathcal{C}$ has the following property. Any function $w : \mathcal{C} \to \mathbb{C}$ on $\mathfrak{G}$ extends to a multiplicative function $(\mathcal{C}^{\otimes}, \sqcup) \xrightarrow{w^{\otimes}} (\mathbb{C}, \cdot)$, and

$$\int_{\mathcal{C}^{\otimes}} w^{\otimes} \;=\; exp(\int_{\mathcal{C}} w).$$

.

3.8.1. *Free tensor categories.* We start with three constructions of the same form:

A. Free monoidal category $C^{\overline{\otimes}}$. Let $C^{\overline{\otimes}}$ be the category whose objects are pairs (A, u) with $A \in \mathcal{OFS}$ and $u \in C^A$, while $\mathrm{Hom}_{C^{\overline{\otimes}}}[(A, u), (B, v)]$ consists of pairs (ι, u) of an isomorphism $\iota : A \to B$ of ordered sets and a map $u \to \iota^* v$ in C^A. We define operation $C^{\overline{\otimes}} \times C^{\overline{\otimes}} \xrightarrow{\sqcup} C^{\overline{\otimes}}$ by $(A, u) \sqcup (B, v) \stackrel{\mathrm{def}}{=} \big(A + B, (u, v)\big)$ (and the induced formula on homomorphisms), where $A + B$ is the *ordered* disjoint union operation in $\mathcal{OFS}$.

We can think of the construction of $C^{\otimes}$ as "spreading of C over $\mathcal{OFS}$" and $\mathcal{OFS} = \mathrm{pt}^{\overline{\otimes}}$. Notice that $C^{\overline{\otimes}}$ contains an equivalent subcategory $C^{\bullet} \stackrel{\mathrm{def}}{=} \sqcup_{n \geq 0} C^n$ (the spreading of C over the full subcategory of $\mathcal{OFS}$ with objects $\langle n \rangle = \{1, ..., n\}$).

B. Free tensor category $C^{\otimes}$. Let $C^{\otimes}$ be the category whose

- objects $Ob(C^{\otimes})$ are pairs (A, u) with $A \in \mathcal{FS}$ and $u \in C^A$ and
- homomorphisms $\mathrm{Hom}_{C^{\otimes}}[(A, u), (B, v)]$ are pairs (ι, u) of a bijection $\iota : A \to B$ and a map $u \to \iota^* v$ in C^A.

Again, we define operation $C^{\otimes} \times C^{\otimes} \xrightarrow{\sqcup} C^{\otimes}$ by $(A, u) \sqcup (B, v) \stackrel{\mathrm{def}}{=} \big(A \sqcup B, (u, v)\big)$. Again, $C^{\otimes}$ contains an equivalent subcategory $\sqcup_{n \in \mathbb{N}} C^n \ltimes S_n$ called the *wreath power of* C. Here, $Ob(C^n \ltimes S_n) = Ob(C)^n$ and

$$\mathrm{Hom}_{C^n \ltimes S_n}[(a_1, ..., a_n), (b_1, ..., b_n)] \;\stackrel{\mathrm{def}}{=}\; \sqcup_{s \in S_n} \prod_1^N \mathrm{Hom}_C(a_i, b_{\sigma i}).$$

C. Free braided monoidal category $E_2(C)$. This is the *braid power* $\sqcup_{n\geq 0}\, C^n \ltimes \mathbb{B}_n$ of C (where $\mathbb{B}_n$ are the braid groups).

Lemma. [Properties of $C^{\otimes}$.] (a) $C\mapsto (C^{\otimes},\sqcup)$ is the left adjoint of the forgetful functor from tensor categories to categories, i.e., for a category C and any tensor category $\mathcal{T}$ there is a canonical equivalence of categories

$$Fun_{\otimes}[(C^{\otimes},\sqcup),\mathcal{T}] \;\cong\; Fun(C,\mathcal{T}).$$

(b) The corresponding functor $C\to C^{\otimes}$ is a full embedding. Actually, $C^{\otimes}$ is a graded tensor category, i.e., a disjoint union $C^{\otimes}=\sqcup_{n\geq 0}\, {C^{\otimes}}_n$ with ${C^{\otimes}}_0=\{0_{C^{\otimes}}\}$ and ${C^{\otimes}}_1=C$.

(c) The above equivalence of categories $Fun_{\otimes}[(C^{\otimes},\sqcup),\mathcal{T}] \;\cong\; Fun(C,\mathcal{T})$ is the composition with $C\to C^{\otimes}$. It has a canonical quasi-inverse $W\mapsto W^{\otimes}$ which extends a functor $W: C\to\mathcal{T}$ to a tensor functor $W:(C^{\otimes},\sqcup)\to(\mathcal{T},\otimes)$ by $W^{\otimes}(\sqcup_{i\in I}\, a_i)=\otimes_{i\in I}\, W(a_i)$ for $a_i\in C,\ i\in I$.

(d) $(_1)$ The only invertible object in $C^{\otimes}$ is the unit. $(_2)$ The isomorphisms classes of C and $C^{\otimes}$ are related by

$$\pi_0(C^{\otimes}) \;\cong\; \mathbb{N}[\pi_0(C)].$$

$(_3)$ $(C^{\otimes},\sqcup)$ satisfies a cancellation law: if $a\sqcup x\cong a\sqcup y$ then $x\cong y$.

Proof. (a) Let $W: C\to\mathcal{T}$ be a functor into a tensor category $(\mathcal{T},\otimes)$. For $(A,u)\in Ob(C^{\otimes})$ let $W^{\otimes}(A,u)\stackrel{\text{def}}{=}\otimes_{a\in A}\, W(u_a)$. This defines a functor $W^{\otimes}:\ C^{\otimes}\to\mathcal{T}$, that takes the morphism $(A,u)\stackrel{(\iota,f)}{\longrightarrow}(B,v)$ to

$$W^{\otimes}(A,u)=\otimes_{a\in A}\, W(u_a)\stackrel{\otimes_{a\in A}f_a}{\longrightarrow}\otimes_{a\in A}\, W(v_{\iota a})=\ \otimes_{b\in B}\, W(v_b)=\ W^{\otimes}(B,v).$$

This is clearly a tensor functor.

(d) $(_2)$ implies $(_3)$. □

Remarks. (0) Similar claims hold for $C^{\overline{\otimes}}$ (here $\pi_0(\mathcal{C}^{\overline{\otimes}})$ is the free monoid generated by the set of isomorphism classes $\pi_0(\mathcal{C})$) and for $E_2(C)$ (here $\pi_0[E_2(C)]\cong\mathbb{N}[\pi_0(C)$ again).

(1) [*The logarithm of a tensor category.*] Let us define *primes* in $(\mathcal{C},\otimes)$ as objects p such that $p\cong a\otimes b$ implies that one of a,b is invertible. We define the logarithm of $(\mathcal{C},\otimes)$ as the subcategory $\log(\mathcal{C})\subseteq\mathcal{C}$ of primes. It carries a free action of the tensor subcategory $(\mathcal{C},\otimes)^*$ of invertible objects. Now we can state:

Corollary. A tensor category $(\mathcal{C},\otimes)$ is free iff (i) it has no invertible objects and (ii) $\log(\mathcal{C},\otimes)^{\otimes}\to(\mathcal{C},\otimes)$ is an equivalence.

Examples. (a) Our prototype of free tensor categories is $\mathcal{FS} = \mathrm{pt}^{\otimes}$. The next example are various classes G of graphs, often the moduli G of graphs in this class is a free tensor category $\mathsf{G} = c\mathsf{G}^{\otimes}$ where $c\mathsf{G}$ is the submoduli of connected nonempty graphs

(b) The free tensor category $\mathcal{FS}^{\otimes}$ on the moduli of finite sets is the moduli Par of partitions ("flower patches") where a partition of a set B is a family of disjoint subset $B_a \subseteq B$ indexed by a's in some finite set A, which covers A. (Notice that some B_a's may be empty.)

In particular for the subcategory $\mathrm{pt} \xrightarrow{\cong} \mathcal{FS}_1 \subseteq \mathcal{FS}$ of one element sets, we have a subcategory $\mathcal{FS} \cong \mathrm{pt}^{\otimes} \subseteq \mathcal{FS}^{\otimes} = \mathsf{Par}$. Also, for the subcategory $\mathcal{FS}_2 \subseteq \mathcal{FS}$ of two element sets, its free tensor category $\mathcal{FS}_2{}^{\otimes} \subseteq \mathcal{FS}^{\otimes} = \mathsf{Par}$ is the moduli category Pai of *pairings* on finite sets ($\stackrel{\text{def}}{=}$ partitions of finite sets into pairs).

Proof. Objects of $\mathcal{FS}^{\otimes}$ are pairs of a finite set A and a functor $\alpha : A \to \mathcal{FS}$. This gives finite sets $L_0 = A$, $L_1 = \sqcup_{a \in A}\, \alpha(a) \stackrel{\text{def}}{=} \{(a,b);\ a \in A \quad \text{and} \quad b \in \alpha(a)\}$ and a map $p : L_1 \to L_0$ by $p(a,b) = a$. (We think of $p : L_1 \to L_0$ as a "forest of depth 1" with L_0 the 0-leaves (= roots) L_1 the 1-leaves (= leaves)

On the other hand a map $p : L_1 \to L_0$ gives a partition of L_1 into disjoint subsets $p^{-1}a,\ a \in L_0$.

(c) [*G-sets*]. For a group G, G-sets, i.e., sets with an action of G, form a free tensor category $(\mathcal{S}et_G, \sqcup) = (c\mathcal{S}et_G)^{\otimes}$ where $c\mathcal{S}et_G$ is the submoduli consisting of nonempty sets with transitive actions.

(d) [$\Bbbk$-*sets*]. For a commutative ring $\Bbbk$ we can consider the moduli of finite $\Bbbk$-sets or of finite etale $\Bbbk$-sets.

(e) *"Group algebras" of tensor categories.* For a tensor category $(\mathcal{C}, \otimes)$, the free tensor category $(\mathcal{C}^{\otimes}, \sqcup)$ has two operations $\sqcup$ and $\otimes$ where for $A^k \in \mathcal{FS}$ one has $(A_1 \xrightarrow{c_1} C) \otimes (A_2 \xrightarrow{c_2} C) = (A_1 \times A_2 \xrightarrow{c_1 \otimes c_2} C)$.

For example, the moduli of partitions $(\mathsf{Pai}, \sqcup)$ is the monoid algebra $(\mathcal{FS}, \sqcup_0)\otimes$ of the tensor category $(\mathcal{FS}, \sqcup_0)$ where $\sqcup_0$ denotes the disjoint union in $\mathcal{FS}$. So Pai carries two operations $\sqcup_0$ ("multiplication") and $\sqcup_1 \stackrel{\text{def}}{=} \sqcup$ ("addition").[26]

3.8.2. *Free tensor categories and exponentiation.* Let C be a stack and $\Bbbk$ a commutative topological ring that contains $\mathbb{Q}$.

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[26] Let us view $\mathcal{FS}^{\otimes}$ as the moduli of maps $L_1 \xrightarrow{p} L_0$, such map indeed gives an L_0 family of finite sets $p^{-1}a,\ a \in L_0$, i.e., an object of $\mathcal{FS}^{\otimes}$. Then $\sqcup$ is the disjoint union operation on maps $(L_1' \xrightarrow{p'} L_0') \sqcup_1 (L_1'' \xrightarrow{p''} L_0'') = (L_1' \sqcup L_1'' \xrightarrow{p' \sqcup p''} L_0' \sqcup L_0'')$ since the operation $\sqcup$ in $\mathcal{FS}^{\otimes}$ glues the L_0'-family of sets and the L_0''-family of sets into one family indexed by $L_0' \sqcup L_0''$. On the other hand $(L_1' \xrightarrow{p'} L_0') \sqcup_0 (L_1'' \xrightarrow{p''} L_0'') = (L_1 \to L_0' \times L_0'')$ for

$$L_1 \ = \ \sqcup_{(a',a'') \in L_0' \times L_0''}\ (p')^{-1}a' \sqcup_0 (p'')^{-1}a'' \ = \ L_1' \times L_0'' \sqcup_0 L_1'' \times L_0'.$$

Lemma. [*Combinatorics of free tensor categories.*] Any function $w : C \to \Bbbk$ extends uniquely to $(C^{\otimes}, \sqcup) \xrightarrow{w^{\otimes}} (\mathfrak{m}, \cdot)$ by $w^{\otimes}(A, u) = \prod_{a \in A} w(u_a)$. Then (provided that $\int_C w$ converges unconditionally)

$$exp(\int_C w) \;\; = \;\; \int_{C^{\otimes}} w^{\otimes}.$$

Proof. Since $\mathbb{N}[\pi_0(C)] \xrightarrow{\cong} \pi_0(C^{\otimes})$ by $n = \sum n_a[a] \mapsto [\sqcup_a \, a^{\sqcup \, n_a}]$, we have

$$\int_{\mathcal{C}^{\otimes}} w^{\otimes} = \sum_{n \in \mathbb{N}[\pi_0 C]} \frac{w^{\otimes}(\sqcup_{a \in \pi_0 C} \, a^{\sqcup \, n_a})}{|\mathrm{Aut}(\sqcup_{a \in \pi_0 C} \, a^{\sqcup \, n_a})|}.$$

To compare with

$$e^{\int_C \, w} = \; exp[\sum_{[a] \in \pi_0(C)} \frac{w(a)}{|\mathrm{Aut}(a)|}] = \prod_{[a] \in \pi_0(C)} \sum_{n_a = 0}^{\infty} \frac{1}{n_a!} (\frac{w(a)}{|\mathrm{Aut}(a)|})^{n_a}$$

$$= \sum_{n \in \mathbb{N}[\pi_0 C]} \prod_{[a] \in \pi_0 C} \frac{w(a)^{n_a}}{|\mathrm{Aut}(a)|^{n_a} \; n_a!} \;\; ;$$

it remains to check that

$$\mathrm{Aut}_{\mathcal{C}^{\otimes}}(\sqcup_{[a] \in \pi_0 \mathcal{C}} \, a^{\sqcup \, n_a}) \;\; \cong \;\; \prod_{[a] \in \pi_0 C} \mathrm{Aut}_{\mathcal{C}}(a)^{n_a} \ltimes S_{n_a}.$$

This is true by definitions. Let $a_1, ..., a_n \in A$ be a list that such that for each $[a] \in \pi_0 \mathcal{C}$ set of indices $\alpha_{[a]} = \{i; \;\; a_i = a\}$ has size n_a (and $\sum_{[a]} \; n_a = n$). $a_1 \sqcup \cdot \cdot \sqcup a_n$ means $(a_1, ..., a_n) \in \mathcal{C}^n$. Recall that $\mathrm{Aut}_{\mathcal{C}^{\otimes}}(a_1, ..., a_n)$ is a disjoint union of all $\Pi_i \mathrm{Aut}_{\mathcal{C}}(a_i, a_\sigma)$ over $\sigma \in S_n$. The σ-contribution to is nonempty iff σ preserves the partition of $\{1, ..., n\}$ into $\alpha_{[a]}, \;\; [a] \in \pi_0 \mathcal{C}, \;\; \text{i.e.,} \;\; \sigma \in \Pi_{[a] \in \pi_0 \mathcal{C}} \; S_{n_a}$. For such σ the contribution is $\prod_{[a] \in \pi_0 \mathcal{C}} \; \mathrm{Aut}(a)^{n_a}$.

Examples. (a) *Localization of exponentiation.* When we consider $t \in \Bbbk$ as a function $\mathrm{pt} \xrightarrow{t} \Bbbk$ then it extends uniquely to $t^{\otimes} : (\mathcal{FS}, \sqcup) = \mathrm{pt}^{\otimes} \to (\Bbbk, \cdot)$ and

$$\int_{\mathcal{FS}} t^{\otimes} \;\; = \;\; e^t.$$

A direct proof is obvious, the LHS is $\sum_{[A] \in \pi_0(\mathcal{FS})} \; \frac{1}{|\mathrm{Aut}(A)|} t^A \;\; , \;\; \text{i.e.,} \;\; \sum_{n \in \mathbb{N}} \; \frac{1}{n!} t^n$.

Lemma. The analogue for the free monoidal category construction is

$$\int_{C^{\overline{\otimes}}} \; w \;\; = \;\; \frac{1}{1 - \int_C \; w} \;\; .$$

Proof. The free monoid $\pi_0(C)^\bullet \overset{\text{def}}{=} \sqcup_0^\infty\ \pi_0(C)^n$ is $\pi_0(C^{\overline{\otimes}})$ via $[a_1]\cdot\cdot\cdot[a_n] \mapsto [a_1 \overline{\otimes} \cdot\cdot\cdot \overline{\otimes} a_n]$. So,

$$\int_{\mathcal{C}^{\overline{\otimes}}} w^{\otimes} = \sum_{n\geq 0} \sum_{([a_1],\ldots,[a_n])\in\pi_0(C)^n} \frac{w^{\overline{\otimes}}(a_1\overline{\otimes}\cdot\cdot\cdot\overline{\otimes}a_n)}{|\mathrm{Aut}_{\mathcal{C}^{\overline{\otimes}}}(a_1\overline{\otimes}\cdot\cdot\cdot\overline{\otimes}a_n)|}$$

$$= \sum_{n\geq 0} \sum_{([a_1],\ldots,[a_n])\in\pi_0(C)^n} \prod_1^n \frac{w(a_i)}{|\mathrm{Aut}_{\mathcal{C}}(a_i)|} = \sum_{n\geq 0} \Big[\sum_{[a]\in\pi_0(C)} \frac{w(a)}{|\mathrm{Aut}_{\mathcal{C}}(a)|}\Big]^n$$

$$= \sum_{n\geq 0} \big(\int_C w\big)^n = \frac{1}{1-\int_C w}.$$

Remark. The analogue for the free braided monoidal category construction would require an interpretation of the size $|\mathbb{B}_n|$ of the braid groups $\mathbb{B}_n$. This may be a series given by some increasing filtration by finite subsets, say $F_p\mathbb{B}_n$ could be the elements written as products of $\leq p$ generators $T_i^{\pm 1}$.

3.8.3. *Fibered products of tensor categories.* The *fibered product* $\mathcal{A}\times_{\mathcal{C}}\mathcal{B}$ of the diagram of categories $\mathcal{A} \overset{\alpha}{\to} \mathcal{C} \overset{\beta}{\leftarrow} \mathcal{B}$ is the category of triples (a, b, ι) where $a \in \mathcal{A},\ b \in \mathcal{B}$ and $\iota : \alpha(a) \cong \beta(b)$ is an isomorphism in $\mathcal{C}$.

Notice that on the level of isomorphism classes of elements, the canonical map

$$\pi_0[\mathcal{A}\times_{\mathcal{C}}\mathcal{B}] \ \to\ \pi_0(\mathcal{A}) \times_{\pi_0(\mathcal{C})} \pi_0(\mathcal{B}).$$

forgets the information of the isomorphism $\iota :\ \alpha(a) \cong \beta(b)$ in $\mathcal{C}$.

Lemma. (a) The category of tensor categories has fibered products. The fibered product of a diagram $(\mathcal{A}, \otimes) \overset{\alpha}{\to} (\mathcal{C}, \otimes) \overset{\beta}{\leftarrow} (\mathcal{B}, \otimes)$, is the fibered product of categories $\mathcal{A}\times_{\mathcal{C}}\mathcal{B}$, with

$$(a, b; \alpha a \underset{\cong}{\overset{\iota}{\to}} \beta b)\otimes(a', b'; \alpha a' \underset{\cong}{\overset{\iota'}{\to}} \beta b') \overset{\text{def}}{=} \big(a\otimes a', b\otimes b'; \alpha(a\otimes a') \underset{\cong}{\overset{\iota\otimes\iota'}{\to}} \beta(b\otimes b')\big)..$$

(b) Invertible objects are given by

$$[(\mathcal{A}, \otimes)\times_{(\mathcal{C},\otimes)}(\mathcal{B}, \otimes)]^* \ \cong\ (\mathcal{A}, \otimes)^*\times_{(\mathcal{C},\otimes)^*}(\mathcal{B}, \otimes)^*.$$

(c) For any diagram of categories $\mathcal{A} \overset{\alpha}{\to} \mathcal{C} \overset{\beta}{\leftarrow} \mathcal{B}$, the fiber at $([a], [b]) \in \pi_0(\mathcal{A}) \times_{\pi_0(\mathcal{C})} \pi_0(\mathcal{B})$ can be described in terms of any chosen base point $\iota :\ \alpha(a) \cong \beta(b)$ as the set

$$\mathrm{Aut}(a)\backslash\backslash\mathrm{Aut}(\alpha(a))// {}^{\iota}\mathrm{Aut}(b).$$

More precisely, for an object $\alpha(a) \underset{\cong}{\overset{\iota}{\to}} \beta(b)$ in $\mathcal{A}\times_{\mathcal{C}}\mathcal{B}$ the fiber $(\mathcal{A}\times_{\mathcal{C}}\mathcal{B} \to \mathcal{A}\times\mathcal{B})_{(a,b)}$ at $(a, b) \in \mathcal{A}\times\mathcal{B}$ (i.e., the moduli of all objects in $\mathcal{A}\times_{\mathcal{C}}\mathcal{B}$ of the form $\alpha(a) \underset{\cong}{\overset{\theta}{\to}} \beta(b)$, is equivalent to the stack $\mathrm{Aut}(a)\backslash\mathrm{Aut}(\alpha(a))/ {}^{\iota}\mathrm{Aut}(b)$.

3.8.4. *Exponential fibered products.* For a tensor category $(\mathcal{C}, \otimes)$, a diagram of categories $\mathcal{A} \xrightarrow{\alpha} \mathcal{C} \xleftarrow{\beta} \mathcal{B}$ defines a diagram of tensor categories $\mathcal{A}^{\otimes} \xrightarrow{\alpha^{\otimes}} \mathcal{C} \xleftarrow{\beta^{\otimes}} \mathcal{B}^{\otimes}$ and then also a tensor category $\mathcal{A}^{\otimes} \times_{\mathcal{C}} \mathcal{B}^{\otimes}$. If $\mathcal{A}^{\otimes} \times_{\mathcal{C}} \mathcal{B}^{\otimes}$ is a free tensor category, we say that

$$\mathcal{A} \diamond_{\mathcal{C}} \mathcal{B} \stackrel{\text{def}}{=} \log(\mathcal{A}^{\otimes} \times_{\mathcal{C}} \mathcal{B}^{\otimes})$$

is the *exponential fibered product* of $\mathcal{A} \xrightarrow{\alpha} (\mathcal{C}, \otimes) \xleftarrow{\beta} \mathcal{B}$.

Example. The moduli of graphs $\mathcal{G}$ is the exponential fibered product of the diagram $\mathcal{FS}_2 \xrightarrow{\subseteq} (\mathcal{FS}, \sqcup) \xleftarrow{id} \mathcal{FS}$.

3.9. **Appendix D. More variations of Feynman expansion.** These examples will not be used.

3.9.1. *One particle irreducible graphs (1PI).* In the case of a partition function Z, the restriction from all graphs $\gamma \in \mathcal{G}$ to connected graphs $\gamma \in c\mathcal{G}$ does not loose any information – we get the logarithm $\log(Z)$.

One can further reduce the the class of graphs to *one particle irreducible graphs* (1PI), meaning the graphs γ which are connected and remain connected when any single edge is removed. Then the integral for the action S over connected graphs is the same as the integral over 1PI graphs of a version S_{eff} of S called the *effective action.*[27]

S_{eff} is a Legendre transform of the original action. The advantage of the effective action is that the correlators for S can be calculated as sums over trees involving the effective action S_{eff}. So, one can view S_{eff} as the fundamental unknown of of a given QFT.

3.9.2. *Colored propagators.* In physics the edges of Feynman graphs often have additional coloring corresponding to a kind of particle they represent.

This just reflects the grading on the space of fields $\mathcal{E} = \oplus_{i \in I} \mathcal{E}_i$. We get $g = \oplus\, g|_{\mathcal{E}_i}$, hence $g^* = \oplus\, (g|_{\mathcal{E}_i})^*$. This leads decomposition of the propagator $P = g^*$ into components $\oplus\, P_i$ means that e^P factors into $\prod\, e^{P_i}$. Then the above computation of $\langle \phi \rangle$ gets refined since a term with an edge labeled by P decomposes into I terms with the same edge labeled by P_i's. We get a generalization of the standard formula in which the edges are colored by I and the amplitudes are calculated in the same way.

Lemma. Let $\mathcal{G}_{I;N}$ be the moduli of graphs with edges colored by I and with N univalent vertices colored by $1, .., N$, then

$$\langle \phi \rangle \;=\; \int_{\gamma \in \mathcal{G}_{I;N}} w^{\gamma}_{P_\bullet}(I, \phi)_{\gamma}.$$

[27] Our central objects are the *effective* or scaled action $S[\alpha]$ corresponding to the scale α. This is a different notion then the 'effective action for the 1PI graphs".

3.9.3. *Matrix integrals.*

4. Feynman integrals in QFT

4.1. Intro.

4.1.1. *Approximations via propagators.* The calculations of Feynman integrals in chapter 3 were done in the case when the space of fields $\mathcal{E}$ is finite dimensional. One way out of this restriction is to use the Feynman expansion formula – a sum of Feynman amplitudes over a moduli of graphs – as a definition of Feynman integrals in general (4.2.1).[28]

The graph definition of integrals applies when graph weights are well defined quantities. As we have observed in 3.5.1. the amplitudes become meaningful when the inverse metric g^* is replaced by any *propagator* P. In the infinite dimensional setting g^* is a priori not a good object[29] and the way to make sense of it is to consider a system of approximations of g^* by propagators.

4.1.2. *Data (M, E, g_M, g_E, I) for a perturbative theory (pQFT).* A perturbative QFT on a manifold M features data of the following type

(1) The vector space of fields $\mathcal{E}$ is the space of smooth sections $\mathcal{E} = C^\infty(M, E)$ of a vector bundle E over M.
(2) The *interaction part* I of the action $S = -\frac{g}{2} + I$ is formal series in fields $I = \sum_0^\infty I_k/k!$ with $I_k \in S^k(\mathcal{E}^*)$.
(3) The quadratic part g of the action is encodeded as an operator Q on $\mathcal{E}$.
One usually works with a metric g_M on the manifold M and a metric $g_E = \langle -, - \rangle$ on the the vector bundle E. This gives a measure dm on M and a metric $g_{\mathcal{E}}$ on sections of E

$$g_{\mathcal{E}}(x, y) \stackrel{\text{def}}{=} \int_M dm \, \langle x(m), y(m) \rangle.$$

Now metric g is represented in terms of $g_{\mathcal{E}}$ by a positive operator Q

$$g(x, y) = g_{\mathcal{E}}(Qx, y), \quad x, y \in \mathcal{E}.$$

In practice Q will be a positive *generalized Laplacian operator*.

[28] However, when the interaction I is not infinitesimal this just an "asymptotic" interpretation of integrals, even in the finite dimensional case.

[29] To transprt g from $\mathcal{E}$ to $\mathcal{E}^*$ we would need $g : \mathcal{E} \to \mathcal{E}^*$ to be an isomorphism. However, the dual of the space $\mathcal{E} = C^\infty(M, E)$ of smooth sections is the space $C^{-\infty}(M, E^!)$ of distributional sections which has some sections that are supported on points.

4.1.3. *Strategy.* The inverse $P = Q^{-1}$ is called the (ideal) *propagator* of the theory. It can be viewed as a distributional section of the outer tensor product $E \otimes E$ on $M \times M$.

The problem that arises is that the above abstract Feynman expansion formalism applies only when the operator P is a smooth section of $E \boxtimes E$. This singular behavior of P can be described as the source of serious infinities in QFT. The solution is to:

(1) Approximate P with nicer operators. We will do it by using the heat kernel operators $K_l = e^{-lQ}$ of Q, which have much better asymptotics then Q or P.
(2) Eliminate infinities by *renormalization procedure*.

We will see that renormalization is a subtle idea which is not well understood in either physics or mathematics. We will apply it in the context of constructing the effective perturbative quantum field theories.

4.1.4. *Philosophical remark: Feynman expansion as a (vague) Feynman measure.* In Feynman's correlator integrals

$$\langle \phi \rangle_S \stackrel{\text{def}}{=} \int_{\mathcal{E}} dx \; e^{S(x)} \phi(x)$$

the basic problem is the meaning of the Feynman measure dx. The separation of the action into two terms $S = -\frac{g}{2} + I$ leads to a Feynman expansion

$$\langle \phi \rangle_{-\frac{g}{2}+I} \stackrel{\text{def}}{=} \int_{\mathcal{E}} dx \; e^{-\frac{g(x)}{2}} \; e^{I(x)} \phi(x) \; = \; \int_{\mathcal{G}_N \ni \gamma} w^{\gamma}_{g^*}(I, \phi).$$

One can view this as the construction of the measure

$$e^{-\frac{g(x)}{2}} dx \stackrel{\text{def}}{=} \int_{\mathcal{G}_\bullet \ni \gamma} w^{\gamma}_{g^*}$$

on $\mathcal{E}$ (with values in $\widehat{S}\mathcal{E}$ so that it can be contracted with $\phi \in S(\mathcal{E}^*)$). This measure corresponds to the free action so it describes what we call "free space" or "empty space".[30] Also, from our point of view I is the basic quantity (rather than $J = e^{I/\hbar}$), this should not be called measure (it is not linear in I) but a composition of a measure with $I \mapsto J$.

Remark. Evidence that Feynman measure should involve the free action: (i) in the finite dimensional case, the normalization of the measure involves the free part of the action; (ii) Wiener measure on path spaces plays the role of $e^{-\frac{g(x)}{2}} dx$ rather than just dx.

4.2. **Infinite dimensional spaces of fields.**

[30] This "empty space" is not really empty as we can borrow virtual particles from it. Moreover, the notion of empty space is not always clear since the splitting $S = -\frac{g}{2} + I$ is not always canonical.

4.2.1. *Topological vector spaces.* The data $(\mathcal{E}, g, I)$ for an abstract Feynman expansion consist of a vector space $\mathcal{E}$, a positive metric g on $\mathcal{E}$ and an infinitesimal function I on $\mathcal{E}$. When $\mathcal{E}$ is finite dimensional we proved that the correlator integral $\langle\phi\rangle_{-\frac{g}{2}+I}$ has an expansion

$$\langle\phi\rangle_{\mathsf{P},I} \stackrel{\text{def}}{=} \int_{\mathcal{G}_N \ni \gamma} w^{\gamma}_{g^*}(I, \phi)$$

where propagator P is chosen to be the dual metric g^* on $\mathcal{E}^*$.

When $\mathcal{E}$ is infinite dimensional we would like to use the above expression $\langle\phi\rangle_{g^*,I}$ as the definition of the correlator $\langle\phi\rangle_{-\frac{g}{2}+I}$. The first observation is that the space of fields is a *topological* vector space.

The calculus on a vector space $\mathcal{E}$ requires a choice of a reasonable topology on $\mathcal{E}$, i.e., a structure of a topological vector space on $\mathcal{E}$ with some good properties. Traditionally one would require that $\mathcal{E}$ be a Hilbert space. However, we will use the class of *nuclear Frechet* topological vector spaces which are more useful for geometry and homological algebra.

Example. The space of smooth functions $C^\infty(M)$ on a manifold M has a nuclear topology given by seminorms $p_{C,D_1,..,D_N}$ – for compact $C \subseteq M$ and differential operators D_i

$$p_{C,D_1,..,D_N}(f) \stackrel{\text{def}}{=} max_{1\leq i\leq N}\ max_{x\in C}\ |(D_i f)(x)|, \quad f \in C^\infty(M).$$

Similarly, for the spaces $C^\infty(M, E)$ of of smooth sections of vector bundles E over M.

Now algebraic operations on vector spaces have to be done in topological vector spaces. For instance $\mathcal{E}^*$ will mean the continuous linear functionals. By definition, $C^\infty(M, E)^*$ is the space of distributional sections $\mathcal{D}(M, E^*)$ of the dual of E.

The tensor product of two topological vector spaces $\mathcal{E}$ and $\mathcal{F}$ presents a richer story. The algebraic tensor product $\mathcal{E}\otimes_{alg}\mathcal{F}$ has a natural topology but what is useful are two completions:

(1) The *projective* completion $\mathcal{E}\otimes_{proj}\mathcal{F}$ of $\mathcal{E}\otimes_{alg}\mathcal{F}$ has the universal property with respect to the continuous bilinear maps $\mathcal{E}\times\mathcal{F} \stackrel{\Phi}{\to} \mathcal{V}$.
(2) The *injective* completion $\mathcal{E}\otimes_{inj}\mathcal{F}$ of $\mathcal{E}\otimes_{alg}\mathcal{F}$ has the universal property with respect to the bilinear maps which are *separately continuous* , i.e., $\Phi(e, -)$ and $\Phi(-, f)$ are continuous for $e \in \mathcal{E}$, $f \in \mathcal{F}$.

We will use the *projective* completion $\mathcal{E}\otimes_{proj}\mathcal{F}$ and we will denote it simply by he *projective* completion $\mathcal{E}\otimes\mathcal{F}$. One of its good properties is that

$$C^\infty(M_1, E_1)\otimes C^\infty(M_2, E_2) \cong C^\infty(M_1\times M_2, E_1\boxtimes E_2).$$

The injective completion may appear sometimes for technical reasons.

4.2.2. *Feynman expansions.* The space of fields $\mathcal{E}$ will be the nuclear space of smooth sections $C^\infty(M,E)$ of a vector bundle E over a manifold M. Now propagators P will be elements of $S^2\mathcal{E}\subseteq\mathcal{E}\otimes\mathcal{E}=C^\infty(M\times M, E\boxtimes E)$. Also, we will require that the homogeneous components I_k of the interaction I lie in $\mathcal{O}^k(\mathcal{E})\stackrel{\text{def}}{=}(\mathcal{E}^{\otimes k})^*]^{S_k}$ so that they can be contracted with elements of $S\mathcal{E}$. Since $\mathcal{E}^{\otimes k})^*=\ [C^\infty(M^k,E^{\boxtimes k}]^*=\mathcal{D}(M^k,(E^*)^{\boxtimes k}]^*$, $\mathcal{O}^k(\mathcal{E})$ are the symmetric distributional sections of $E^*\boxtimes\cdot\cdot\boxtimes E^*$ on M^k.

Remark. The key problem of applying the abstract Feynman expansion to QFT is that the inverse metric g^* is a distributional section of $E\boxtimes E$ which is *not* smooth. This makes amplitudes in Feynman's expansion infinite.

So, we have started with a "huge" Feynman integral and replaced it by the formal Feynman expansion, however, the terms in this expansion are still infinite. One fights this problem by regularizing g^*, i.e., approximating g^* by propagators $P\in S^2\mathcal{E}$ which are smooth sections. Then $\langle\phi\rangle_{P;I}\stackrel{\text{def}}{=}\int_{\mathcal{G}_N} w_P^\gamma(I,\phi)$ is well defined but its limit as P approaches g^* is still infinite. This is resolved by the renormalization procedure which "subtracts from I the part that produces infinities". This can be made to make sense, however the notion of "the part that produces infinities" is not uniquely defined so one needs to control the choices that occur.

4.3. **Approximations of the dual metric.** There are two problems:

(1) The dual metric g^* on $\mathcal{E}^*$ is not a good object in infinite dimension and even formally it leads divergent weight integrals.
(2) In physics the quadratic part $g(x)=(Qx,x)$ is not quite a metric – it has a kernel, the classical solutions of the free theory. Since g^* was defined as the dual metric of g, it does not quite have a sense.

A way to deal with (2) is to drop the kernel of Q from Feynman integrals, i.e., roughly to integrate over $\mathcal{E}/\mathrm{Ker}(Q)\cong\ Im(Q)$.[31] This will be systematically pursued below in the derived version of QFT (the Batalin-Vilkovisky formalism).

Our approach to problem (1) is to approximate g^* by propagators $\mathsf{P}\in S^2\mathcal{E}$ so that the graph weights are well defined. Then we think of g^* as the "ideal propagator" while we effectively work with its truncations P.

We outline two strategies for this approximation ("regularization"). Both are based on good properties of the generalized Laplace operator Q which represents the metric g on fields. The first one ("energy scale") uses the spectral decomposition of fields with respect to Q. The second ("length scale") is based on the heat kernel of Q.[32]

[31] One may still want to remember that $\mathrm{Ker}(Q)$ contributes an infinite factor $e^{Vol(\mathrm{Ker}(Q)}$ to the Feynman integral.

[32] In this approach one uses the formula $a^{-1}=\int_0^\infty dl\ e^{-la}$ – if g is represented by an operator a then g^* is represented by a^{-1}. The right hand side makes sense even if a is not invertible, i.e., if g is degenerate and this will be our replacement for g^* :

While "energy" approach is natural from the point of view of physics, the advantage of the "length" picture is that it local. In the remainder we will adopt the length view which is more geometric (mathematics) and allows better understanding of the Locality Principle (physics) in the effective formalism of QFT.

Below we assume that M is compact. This is actually a prototype of the general strategy – for noncompact M we will cope by switching attention to approximations that have the same kind of asymptotics as in the compact case.

4.3.1. *Generalized Laplacian operators.* We normalize the Laplacian operator Δ on a Riemannian manifold (M, g_M) as

$$\Delta \stackrel{\text{def}}{=} -g_M^*$$

(the dual metric g_M^* is a section of $S^2\mathcal{T}_M$ a subsheaf of the sheaf of differential operators).

If g_M is flat locally there are commuting orthonormal vector fields ∂_i and then $\Delta = -\sum \partial_i^2$.

If M is compact then $\Delta \geq 0$ with a unique 0 eigenvector 1_M. For instance on a flat torus $M = \mathbb{R}^n/\mathbb{Z}^n$, any $\lambda \in \mathbb{Z}^n$ defines an eigenvector $e^{i\lambda}$ with $\partial_k e^{i\lambda} = i\lambda_k e^{i\lambda}$, hence $\Delta e^{i\lambda} = (\sum \lambda_k^2)\, e^{i\lambda}$.

A generalized Laplacian operator on a vector bundle E is a degree 2 differential operator Q whose symbol $\boldsymbol{\sigma}(Q) \in \Gamma[T^*M,\ \mathrm{End}(E)]$ is $\boldsymbol{\sigma}(Q) = id_E \otimes g_M^*$ for a metric $g_M \in C^\infty(S^2T^*M)$.

4.3.2. *The energy view: spectral analysis of Q.* From the point of view of spectral analysis, it is natural to replace our space of fields $\mathcal{E} = C^\infty(M, E)$ with its thicker version $\underline{\mathcal{E}} \stackrel{\text{def}}{=} L^2(M, dm; E)$. So, $\mathcal{E} = C^\infty(M, E)$ lies in its Hilbert completion $L^2(M, dm; E)$ while the dual Hilbert space $L^2(M, dm; E^*)$ lies in the distributional sections $\mathcal{D}(M, E) = C^\infty(M, E)^* = \mathcal{E}^*$. The basic fact is that

Lemma. On a compact manifold M any generalized Laplace operator Q has a discrete spectral decomposition $L^2(M, dm; E) = \oplus_0^\infty \mathbb{C}e_i$ such that the eigenvalues $Qe_i = \lambda_i e_i$ can be organized as $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ with finite multiplicities and $\lambda_i \to \infty$. The eigenvectors e_i are smooth functions (analytic when M is) and can be chosen orthonormal.

For a semisimple operator A on a finite dimensional vector space W, if $W = W_0 \oplus W'$ is the decomposition into the zero eigenspace and the rest then

$$\int_\varepsilon^L dl\ e^{-lA} = 0_{W_0} \oplus (\int_\varepsilon^L dl\ e^{-lA|_{W'}}.$$

Remarks. (0) On $\underline{\mathcal{E}} = L^2(M, dm; E)$ the metric g is represented by the (densely defined) differential operator Q, so the dual metric g^* on $L^2(M, dm; E^*)$ is represented by the inverse adjoint operator Q^{-*}.[33] When we identify $L^2(M, dm; E)$ with its dual then the operator that represents g^* is just Q^{-1}.

(1) In terms of $e^i = g_{\mathcal{E}}(e_i, -)$, one can write $g = \sum \lambda_i e^i \otimes e^i$ and $g^* = \sum {\lambda_i}^{-1} e_i \otimes e_i$ (also $Q = \sum \lambda_i e_i \otimes e^i$ and $Q^{-1} = \sum {\lambda_i}^{-1} e_i \otimes e^i$). So, g^* clearly lies in some completion of $\mathcal{E} \otimes_{alg} \mathcal{E}$. The problem is that it does not lie in the projective completion $\mathcal{E} \otimes \mathcal{E} = C^\infty(M^2, E \boxtimes E)$.

4.3.3. *Spectral approximations.* We will use symbol Λ for the *energy* scale (by energy we mean the eigenvalues of the "kinetic" operator Q). The Hilbert space $\underline{\mathcal{E}} = L^2(M, dm; E)$ decomposes into $\underline{\mathcal{E}}_{\leq\Lambda} \oplus \underline{\mathcal{E}}_{>\Lambda}$ where $\underline{\mathcal{E}}_{\leq\Lambda}$ is finite dimensional. We approximate the incarnation Q^{-1} of the dual metric g^* by its $\underline{\mathcal{E}}_{\leq\Lambda}$ component

$$P_{\leq\Lambda} \stackrel{\text{def}}{=} \sum\nolimits_{-} \lambda_i \leq \Lambda \ {\lambda_i}^{-1} \ e_i \otimes e^i.$$

This kind of truncation is natural in physics since the energy that one can use in any given experiment is bounded, so so in this sense at any time we are observing the $\leq \Lambda$ truncation of the physical system.

4.3.4. *The length view: heat kernel of Q.* While $Q^{-1} e_i = {\lambda_i}^{-1} e_i$ (with ${\lambda_i}^{-1} \to 0$) has a better asymptotics than Q we know that this is not good enough. However, operators e^{-lQ} for $l > 0$ have much better asymptotics and one can recover Q^{-1} by

$$\int_0^\infty dl \ e^{-lQ} \ = \ [e^{-lQ}(-Q)^{-1}]_{l=0}^\infty \ = \ Q^{-1}.$$

Theorem. On a compact M, the operators e^{-lQ} for $l \geq 0$ have a distributional integral kernel $K_l \in \mathcal{D}(M \times M, E \boxtimes E)$. Moreover

(1) For $l > 0$, K_l is a smooth function (analytic when M is).
(2) $K_0(x, y) = \delta(x - y) = \delta_{\Delta_M}$ is the diagonal delta distribution. (The x-family of distributions $\delta_x(y)$ on M.)

Remark. Here Q^{-1} is an integral operator with a distributional kernel $Q^{-1}(x, y) = \int_0^\infty dl \ K_l(x, y) \in \mathcal{D}(M^2, E \boxtimes E)$. This is a way of making sense of g^* as a distributional section of $E \boxtimes E$.

[33] Since g is nondegenerate we have $Q > 0$, i.e., $\lambda_i > 0$. Notice that this is not true for the scalar Laplacian operator $-\sum \partial_i^2$, so in the scalar theory one actually uses its "mass regularization" $Q = \Delta + m$ with positive mass m.

4.3.5. *Length approximations.* We will use symbols ε, L for the "*length*" scale. We approximate the incarnation of g^* by the integral kernel $Q^{-1}(x,y) = \int_0^\infty dl\ K_l(x,y) \in \mathcal{D}(M^2, E\boxtimes E)$ with the integral kernel which is a finite integral

$$P_{\varepsilon L}(x,y) \stackrel{\text{def}}{=} \int_\varepsilon^L dl\ K_l(x,y).$$

The properties of the heat kernel (theorem 4.3.4) guarantee that this is a smooth section of $E\boxtimes E$, i.e., a propagator $P_{\varepsilon L} \in S^2\mathcal{E}$.

The singularities that appear as $L \to \infty$ are called *infrared* and the one for $\varepsilon \to 0$ are called *ultraviolet*. Infrared singularities appear for noncompact M and will be easier to deal with. The energy Λ and length ε are reciprocal ("probing at small lengths requires large energy"), so the ultraviolet regime corresponds to the large energy singularities in the energy picture. These are the singularities that will require renormalization.

Mathematically it is quite natural to use the beautiful theory of heat kernels for truncation of g^*, i.e., Q^{-1}. The physical interpretation is more controversial. It interprets the "length" l as the ("proper") time that particle travels between two interactions. However, the particles that appear in this picture are "nonphysical" ("virtual") in the sense that they violet some standard rules. For instance they cam move forward and backward in time.

So, one may want to think that the unobserved virtual particles are a part of nature or one may say that they just appear – formally and mysteriously – in Feynman's rules for calculating correlators.

4.3.6. *Translation between energy and length.* Analytically, the energy, i.e., spectral analysis regularization uses a sharp cutoff, while the length regularization is technically better since it uses a smooth cutoff. (The energy cutoff $P_{\Lambda',\Lambda}$ of the "ideal propagator" Q^{-1} jumps at eigenvalues while $P_{\varepsilon L}$ is smooth in ε, L.

In terms of $Q = \sum_i \lambda_i\ e_i \otimes e^i$ we have

$$P_{\varepsilon L} = \int_\varepsilon^L dl\ e^{-lQ} = \int_\varepsilon^L dl\ \sum_i e^{-l\lambda_i}\ e_i\otimes e^i = \sum_i \frac{e^{-L\lambda_i} - e^{-\varepsilon\lambda_i}}{\lambda_i}\ e_i\otimes e^i.$$

So, the high modes $\lambda_i > \Lambda$ are not discarded, they are just suppressed by exponential factors $e^{-a\lambda_i}$ for positive a's.

Abstractly the two formalisms are equivalent in a sense that one can translate between the two pictures. One expresses the heat kernel RGF $P_{\varepsilon L}$ in terms of eigenvectors as above and then in terms of the energy RGF $P_{\Lambda',\Lambda}$. (This is the usual Fourier transform between the position quantities and momentum quantities.)

4.4. **Graph weights as integrals and their ultraviolet singularities.** We consider the ultraviolet behavior, i.e., the $\varepsilon \to 0$ regime of the weights $w^\gamma_{\varepsilon,L} I$. The γ-weight is itself

an integral over a power M^{V_γ} of M given by vertices of γ. Its asymptotics is controlled by a neighborhood of the diagonals in M^{V_γ}, for that reason it does not essentially depend on M.

We will see that the singularity, i.e., the divergence of the limit $\lim_{\varepsilon\to 0}\ w^\gamma_{\varepsilon,L}(I)$, is caused the by loops in the graph γ. From the point of view of analysis while the computations without loops are based on outer product of distributions $\mathcal{D}(M)^{\otimes n} \to \mathcal{D}(M^n)$, the loops use the product of distributions on a single manifold $\mathcal{D}(M)^{\otimes 2} \to \mathcal{D}(M)$, and this is not a priori defined in analysis. So, while the outer product of distributions is defined, the inner product is not, so it has to be renormalized!

The point is that each edge contributes a propagator $P_{\varepsilon L}$ with integral kernel $P_{\varepsilon L}(u,v) = \int_\varepsilon^L\ dl\ K_l(u,v),\quad (u,v\in M)$, and the important part $K_\varepsilon(u,v)$ converges for $\varepsilon\to 0$ to the delta distribution $\delta_{u=v}$ on $\Delta_M \subseteq M^2$. Then a chain of simple edges between non-repeating vertices $1,...,n$ (an A_n-graph) contributes the convolution of δ-distributions $\delta_{u_1=u_2}\cdots\delta_{u_{n-1}=u_n}$ which is a well defined distribution on M^n – the delta distribution on the diagonal $\Delta_M\subseteq M^n$. Geometrically, we got Δ_M out of M^n by imposing $n-1$ equalities between n factors.

On the other hand a loop, i.e., a chain of simple edges between vertices $1,...,n$ which form a cycle, i.e., n coincides with 1 (this is an A^{affine}_{n-1}-graph), requires imposing n equalities between n factors: $u_1=\cdots=u_n=u_1$.

Below we will repeat these arguments with more details.

4.4.1. *Graph weights as integrals.* For simplicity we will calculate in a scalar theory so that $\mathcal{E}=C^\infty(M)$, in this case the contractions of tensors reduce to products of functions. We assume that M is compact, so the measure dm on M (given by the metric g_M) is a linear functional $\phi\in\mathcal{E}^*$. The interaction is a formal series $I=\ \sum_k\ \frac{1}{k!}I_k$, with I_k a *local* functional of degree k, i.e., a finite sum of terms of the form

$$\mathcal{E}\ni x\mapsto \int_M\ dm\ \prod_1^k\ (D_i x)(m)$$

where D_i are differential operators. For simplicity we choose a representative case when $D_i=1$ and $I_k(x)=\int_M\ dm\ x(m)^k$.

Compactness also gives for the operator Q the canonical heat kernel K. We will use propagators $P_{\varepsilon L}=\int_\varepsilon^L\ dl\ e^{-lQ}=\ \int_\varepsilon^L\ dl\ K_l$.

Lemma. (a) For any propagator P the weight $w^\gamma_P(I)$ viewed as a function of an infinitesimal field a is

$$w^\gamma_P(I)\ (a)\ =\ \int_{M^{V_\gamma}}\ \prod_{v\in V_\gamma}\ du_v\quad \prod_{v\in V_\gamma}\ a(u_v)^{t_v}\ \prod_{e\in E_\gamma}\ P(u_{e'},u_{e''}).$$

Here V_γ, E_γ are vertices and edges of γ and the ends of an edge e are e',e''.

(b) If $P = P_{\varepsilon L}$ the formula for $w^\gamma_{\varepsilon L} \stackrel{\text{def}}{=} w^\gamma_{P_{\varepsilon L}}$ has more structure

$$w^\gamma_{\varepsilon L}(I)\ (a)\ =\ \int_{M^{V_\gamma}} \prod_{v\in V_\gamma} du_v \quad \int_{[\varepsilon,L]^{E_\gamma}} \prod_{e\in E_\gamma} dl_e \qquad \prod_{v\in V_\gamma} a(u_v)^{t_v} \quad \prod_{e\in E_\gamma} K_{l_e}(u_{e'}, u_{e''}).$$

Proof. For any propagator $P \in S^2\mathcal{E} = C^\infty(M^2)^{S_2}$ the meaning of the contraction with a product $\phi\otimes\psi$ of $\phi, \psi \in \mathcal{E}^* = C^{-\infty}(\mathcal{E})$ is

$$\langle P, \phi\otimes\psi\rangle\ =\ \int_{M^2} dm_1\ dm_2\ \phi(m_1)P(m_1,m_2)\phi(m_2).$$

□

Below, we will write this in examples.

4.4.2. *Star graphs.* Let γ have one vertex with k tails and no edges. Then I_γ is just I_k and $P_\gamma = 1$ (no edges) hence there is no ε-dependence. If we put fields $a_1, ..., a_m$ at the tails then

$$w^\gamma(I)\ (a_1\otimes\cdots\otimes a_k)\ =\ \langle P_\gamma\otimes a_1\otimes\cdots\otimes a_k, I_\gamma\rangle_\gamma\ =\ \int_M du\ a_1(u)\cdots a_k(u).$$

4.4.3. *Ladder graphs.* Let γ be a graph with tails such that the underlying graph has vertices $1, .., n$ and edges $\boxed{12}, ..., \boxed{n-1, n}$ (the A_n-graph). Let t_i and k_i be the number of tails and valency at the vertex i. Then $P_\gamma = P^{\otimes\ E_\gamma} = P^{\otimes\ n-1}$ and $I_\gamma = \otimes_{v\in V_\gamma}\ I_{k_v} = \otimes i = 1^n\ I_{k_i}$. The weight $w^\gamma_P(I) = \langle P_\gamma, I_\gamma\rangle_\gamma$ is a homogeneous polynomial in an infinitesimal field a of degree $\sum\ t_i$,

$$w^\gamma_P(I)\ (a)\ =\ \int_{M^n} dm_1\cdots du_n\ \ a(u_1)^{t_1}P(u_1,u_2)\cdots a(u_{n-1})^{t_{n-1}}P(u_{n-1},u_n)a(u_n)^{t_n}$$

Now, if $P = P_{\varepsilon L} = \int_\varepsilon^L\ dl\ K_l$, this is

$$\int_\varepsilon^L dl_1\cdots\int_\varepsilon^L dl_{n-1}\ \int_{M^n} du_1\cdots du_n\ \ a(u_1)^{t_1}K_{l_1}(u_1,u_2)\cdots a(u_{n-1})^{t_{n-1}}K_{l_{n-1}}(u_{n-1},u_n)a(u_n)^{t_n}.$$

4.4.4. *Loops.* Now let the above chain be a loop, i.e., we ask that two vertices coincide: $n = 1$ (hence $t_n = t_1$). Then there are only $n-1$ vertices, hence $n-1$ integrals over M, but there are still $n-1$ edges. So, $w^\gamma_{\varepsilon L}(I)\ (a)$ is

$$\int_\varepsilon^L dl_1\cdots\int_\varepsilon^L dl_{n-1}\ \int_{M^{n-1}} du_1\cdots du_{n-1}\ \ a(u_1)^{t_1}K_{l_1}(u_1,u_2)\cdots a(u_{n-1})^{t_{n-1}}K_{l_{n-1}}(u_{n-1},u_1).$$

4.4.5. *Divergence.* We consider the $\varepsilon \to 0$ behavior of the above graph integrals. When $\varepsilon \to 0$ then the integral kernel $K_\varepsilon(x,y)\ dy$ approaches the delta distribution $\delta(x-y)$ on $\Delta_M \subseteq M^2$, i.e., $\lim_{\varepsilon\to 0}\ \int_M\ K_\varepsilon(u,v)\ x(v)\ =\ x(u)$.

The $\varepsilon \to 0$ behavior of the ladder graph integral is controlled by the quantity

$$\int_{M^n}\ du_1\cdots du_n\ \ a(u_1)^{t_1}K_\varepsilon(u_1,u_2)\cdots a(u_{n-1})^{t_{n-1}}K_\varepsilon(u_{n-1},u_n)a(u_n)^{t_n}$$

which approaches

$$\int_{M^n}\ du_1\cdots du_n\ \ a(u_1)^{t_1}\delta(u_1,u_2)\cdots a(u_{n-1})^{t_{n-1}}\delta(u_{n-1},u_n)a(u_n)^{t_n}\ =\ \int_M\ du\ a(u)^{t_1+\cdots+t_n}.$$

For this reason $\lim_{\varepsilon\to 0}\ w_{\varepsilon L}^\gamma(I)$ does exist for ladder graphs.

On the other hand for the loop integral, the corresponding quantity

$$\int_{M^{n-1}}\ du_1\cdots du_{n-1}\ \ a(u_1)^{t_1}K_\varepsilon(u_1,u_2)\cdots a(u_{n-1})^{t_{n-1}}K_\varepsilon(u_{n-1},u_1)$$

approaches something like

$$\int_{M^{n-1}}\ du_1\cdots du_n\ \ a(u_1)^{t_1}\delta(u_1,u_2)\cdots a(u_{n-1})^{t_{n-1}}\delta(u_{n-1},u_1)=\ \int_M\ du\ a(u)^{t_1+\cdots+t_n}\delta(u,u).$$

However for the delta distribution $\delta(u,v)\ =\ \delta(v-1)$ one has $\delta(u,u)\ =\ \infty$, hence $\lim_{\varepsilon\to 0}\ w_{\varepsilon L}^\gamma(I)$ diverges.

The conclusion is that (as stated above), the divergences are caused by loops and more loops means worse divergence. In other words,

Corollary. No divergence happens precisely for trees.

4.4.6. *Asymptotics of heat kernels.* A more quantitative examination of the asymptotics of weight integrals is again based on precise asymptotics of heat kernels.

Proposition. For $\varepsilon \to 0$, the asymptotics of the heat kernel on a manifold (M, g_M) is (in terms of the g_M-distance d_M),

$$K_\varepsilon(u,v)\ \sim\ \frac{e^{-d_M(u,y)^2/2\varepsilon}}{2\pi\varepsilon^{\dim(M)/2}}\ dv.$$

Remark. Now the nonexistence of the self multiplication of $\delta_{u=v}$ is seen as the absence of the $\varepsilon \to 0$ limit of $K_\varepsilon(u,v)^2\ \sim\ \frac{e^{-d_M(u,y)^2/\varepsilon}}{2\pi\varepsilon^{\dim(M)}}\ dv$.

4.5. **"Particle" interpretation of Feynman's graph expansions.** In QFT Feynman's graph expansion formula has interpretation as an integral over the moduli of maps from metrized graphs into M.

First, locality of action means that each homogeneous piece I_k of the interaction I is an integral over M. This represents a graph weight $w_P^\gamma(I)$ as an integral over the space M^{V_γ} of "γ-vertices in M", i.e., maps of V_γ into M (at each vertex $v \in V_\gamma$ there is one I_k, hence one integral over M).

On the other hand the formal identity $Q^{-1} = \int_0^\infty \, dl \; e^{-lQ}$ (for the "ideal" propagator $P = Q^{-1}$) can be used to presentation $w_P^\gamma(I)$ as in integral over the space $Met(\gamma)$ of all "metrics" l on the graph γ. By a metric l on γ we mean assigning to each edge $e \in E_\gamma$ a "length" $l_e \geq 0$, and each edge e contributes one integral $\int_0^\infty \, dl_e$.

Finally, the interpretation of the heat kernel $K_l = e^{-lQ}$ as a Wiener integral can be used to refine the above representation of $w_P^\gamma(I)$ as in integral over all maps of the geometric realization $|\gamma|$ into M. Here, we write the heat kernel

$$K_l(x,y) \;=\; \int_{f:[0,l]\to M} 1 \; d\mathcal{W}_l \;=\; \int_{f:[0,l]\to M} df \; e^{-E_l(f)}$$

first in terms of the Wiener measure $\mathcal{W}_l$ o the space of paths of length l, and then as a Feynman integral with respect to a Feynman measure df of the energy action of the path f,[34]

All together, the space over which one integrates has the following variables: (i) a metric l on γ (from $Q^{-1} = \int \, dl \; e^{-lQ}$), (ii) a map $f_e : [0, l_e] \to M$ which at the end points of the interval agrees with the map on vertices: $f_e(0) = f(e')$ and $f_e(l_e) = f(e'')$. So, we are integrating over the moduli of maps of metrized versions of the geometric realization $|\gamma|$ into M.

4.5.1. *Heat kernel as a Wiener integral.* Let $\mathcal{W}_l$ (resp. $W_l(x,y)$) be the Wiener measure on the space $\mathcal{P}_l(M)$ of continuous paths in M of length l, i.e., maps $[0,l] \xrightarrow{f} M$ (resp. the space $\mathcal{P}_l(M;x,y)$ of f with $f(0) = x$ and $f(l) = y$).

Lemma. The heat kernel is the volume of $\mathcal{P}_l(M)$ for the Wiener measure

$$K_l(x,y) \;=\; \int_{f\in\mathcal{P}_l(M)} 1 \; d\mathcal{W}_l(x,y) \; (f)$$

[34]Feynman integrals are defined for Quantum Mechanics, i.e., the 1-dimensional case of QFT. So, the only precise meaning of the Feynman integral (the second integral), is the Wiener integral (the first integral), i.e., the Feynman measure is

$$df \stackrel{\text{def}}{=} e^{E_l(f)} \; d\mathcal{W}_l f.$$

Remark. One can interpret this formula as a Feynman integral where fields are paths and the action is the energy action $E(f)$ of a path f

$$K_l(x,y) \;=\; \int_{\mathcal{P}_l(M)} 1\; d\mathcal{W}_l(x,y) \;=\; \int_{\mathcal{P}_l(x,y)} df\; e^{-E_l(f)}.$$

This is in fact the one case where Feynman integrals are well defined – the 1-dimensional QFT ("Quantum Mechanics"). The precise meaning of the Feynman integral (the second integral) is just the Wiener integral (the first integral). So, the the Feynman measure is

$$df \stackrel{\text{def}}{=} e^{E_l(f)}\; d\mathcal{W}_l(f).$$

4.5.2. *Metrized graphs in a manifold M.* For any propagator P the weight $w_P^\gamma(I)$ viewed as a function of a field a is an integral (see lemma 4.4.1.a)

$$w_P^\gamma(I)\;(a) \;=\; \int_{M^{V_\gamma}} \prod_{v\in V_\gamma} du_v \quad \prod_{v\in V_\gamma} a(u_v)^{t_v} \quad \prod_{e\in E_\gamma} P(u_{e'}, u_{e''})$$

over the space M^{V_γ} of "vertices of γ in M", i.e., maps from vertices V_γ to M.

In the heat kernel approach we consider the idealized propagator of the theory P with its localization on the length scale: $P = \int_0^\infty dl\; e^{-lQ} = \int_0^\infty dl\; K_l$. This introduction of the length scale gives a richer structure which will extend the appearance of γ-vertices in M to an appearance of the whole graph γ in M (moreover, γ comes with an internal metric).

We write the weight formula for the truncated version of the propagator: $P_{\varepsilon L} = \int_\varepsilon^L dl\; K_l$ (lemma 4.4.1.b):

$$w_{\varepsilon L}^\gamma(I)\;(a) \;=\; \int_{M^{V_\gamma}} \prod_{v\in V_\gamma} du_v \quad \int_{[\varepsilon,L]^{E_\gamma}} \prod_{e\in E_\gamma} dl_e \quad \prod_{v\in V_\gamma} a(u_v)^{t_v} \quad \prod_{e\in E_\gamma} K_{l_e}(u_{e'}, u_{e''}).$$

The space $[\varepsilon, L]^{E_\gamma}$ is the moduli $Met(\gamma)$ of "metrics on the graph γ" where a metric is a choice of "lengths" of edges. A geometric realization of a metrized graph (γ, l) is a geometric realization of γ with a metric on the legs such that the leg corresponding to edge e has length l_e (it is determined by (γ, l) up to a unique isomorphism).

This is one of the factors. For the other one recall that we are integrating over the space M^{V_γ} of maps $V_\gamma \xrightarrow{f} M$. Moreover, each heat kernel factor K_{l_e} (with $e \in E_\gamma$)) in the integrand, has a representation as the Wiener integral over maps f_e of the interval $[0, l_e]$ into M such that it sends the ends e', e'' of the edge e to $f(e')$ and $f(e'')$.

4.5.3. *Particle interpretation.* We can view a metrized graph as a worldline of a finite systems of particles which carry an "internal clock" (called *proper time*) and are allowed to collide and break apart. Then the maps into M can be thought of as all possible evolutions of systems of particles in spacetime M.

So, the graph expansion of a quantum theory of fields takes form of a quantum theory of particles in the sense that the Feynman graphs in the expansion get interpreted as world-lines of systems of particles. Therefore, in its heat kernel incarnation, i.e., the geometric incarnation, perturbative expansion appears as a passage "from fields to particles".

The particle interpretation (i.e., calling the maps from metrized graphs to M particles) is controversial because these particles violate various properties that known particles have. For instance, if M is the spacetime then the inner time on a particle ("proper time") is unrelated to the time in M, so these particles can move backwards and forwards in time. An example of how the Feynman graph particles violate conservation of momentum is sketched in 5.10.

On the other hand, I understand that this particle picture is the same as the one that appears for Feynman expansions in the operator formalism.

Remark. Decomposing interaction terms into sums $I_k = \sum_{\alpha\in A_k} I_{k,\alpha}$ leads to ra finer picture, i.e., a Feynman expansion with more summands as one refines the class of graphs (by adding colors A_k at k-valent vertices). So, any interpretation of the graph expansion should also account for this mechanism.

4.5.4. *Singular Quantum Mechanics.* The graph expansion of a QFT on M is a kind of 1-dimensional QFT whose fields "metrized graphs in M" in the sense of maps $f : (\gamma, d) \to M$, of metrized graphs (γ, d) into M. The quadratic part of the action for the new theory is the kinetic energy action on maps $f : (\gamma, d) \to M$,

$$E(f) = \int_{(\gamma,g)} |df|^2/2.$$

The interaction part of the new theory comes from the interaction I of the original QFT.

In the standard terminology QM is a one dimensional QFT but without interactions, i.e., only paths appear but not graphs. So, the above formalism is a kind of a "singular QM" – the worldsheets are now allowed to degenerate from intervals to their singular version – the graphs.

4.5.5. *Relation to String theory.* The asymptotic expansion of QFT in terms of metrized graphs on M is parallel to the idea that the String Theory is an asymptotic expansion of the M-theory whose fields are maps of Riemann surfaces (worldlines of particles that are loops rather than points) into the spacetime M. In fact, the graph expansion should be a little part of the string expansion: when the loop contracts to a point, i.e., a string particle to a classical particle, then the Riemann surface contracts to a (hopefully metrized) graph.

4.5.6. *Questions.* (0) Costello's formalism uses only the stable Feynman graphs, i.e., the ones that correspond to stable surfaces that appear in the Gromov-Witten theory.

(1) The physical meaning of this transition seems to be that all mechanisms in QFT are mediated by particles?

(2) How is the appearance of singular intervals (graphs) parallel to the need in String Theory to use the compactification of surfaces by "stable surfaces"?

Part 2. Quantization by renormalization

Appendices B,C,D are not needed in the remainder.

4.6. Intro.

4.6.1. *The notion of effective quantum theories: physics.* The strategy called *effective* or *scaled* QFT is that instead of considering one setting which involves all possible scales one has a family of "pictures" – for each scale α the α-picture considers only the phenomena of scale $\leq \alpha$.

This approach views the description of a quantum system by a single quantum action $S \in \mathcal{A}ct_q$ as an approximation which produces infinities in QFT. Rather,

(Wilson) *Physics depends on the scale and at each scale α it is described by an action S_α.*[35]

We call $S_\alpha = S[\alpha]$ the "effective" action on scale Λ.

So, the point of view is that in practice one always deals with objects of a limited "size" and then our attempt to describe physics at all scales by a single formula is an idealization which introduces infinities. The infinities only arise when we think of the classical action S as adequate for all energy scales including the infinite one.

Now, the problem of quantization is to deduce $S[\Lambda]$ from S (as much as one can deduce the quantum system from knowing the classical one).

4.6.2. *Change of scale as "coarse graining".* The obvious reason that we require different theories is the "coarse graining" or "defocusing" of degrees of freedom. In studying physics we are generally not interested in knowing all about the world but only about the dominant effects at the scale that we concentrate on. The objects on a smaller scale are not relevant themselves but only through their aggregate effect on the scale that we consider.

This change of scale from α to β through "coarse graining", i.e., averaging the degrees of freedom that live on intermediate scales, is called the *Renormalization Group Flow*. We

[35] How physics depends on the scale is clear when we consider drastically different scales – these require drastically different theories:

(1) Condensed matter
(2) Nuclear
(3) Particle physics.
(4) ? Strings or M-theory
(5) ?

Here (1) lives at standard energies, say the room temperature. To get from (1) to (2) we need to increase energy by a 10^6 factor. From (2) to (3) by a 10^3 factor. We only understand well the physics of "low" energies.

can view it as flow W in the space of all possible actions, then the flow $W_{\alpha\beta}$ moves the description S_β of the system at scale β to S_α. The equation

$$W_{\alpha\beta}S_\beta \;=\; S_\alpha$$

is called the *Renormalization Group Equation* (RGE).

4.6.3. *Effective QFT as regularization by "scaling".* Technically, the idea of *effective* QFT is to regularize a given theory $\mathcal{T}$ by "*scaling*" the theory $\mathcal{T}$ according to some choice of a scale $\mathcal{S}$. This vague notion of scaling means that for each value α of $\mathcal{S}$ the truncated theory $\mathcal{T}_{\leq\alpha}$ considers objects of scale $\leq \alpha$ and their physics is described by an action S_α.

4.6.4. *Renormalization Group Flow.* The compatibility between different scales α, β is then described by the *scaling flow* operators (the *Renormalization Group Flow* operators) $W_{\alpha\beta}$ on the space of actions – one requires that

$$S_\beta \;=\; W_{\alpha\beta}S_\alpha.$$

The idea is that for $\beta < \alpha$ one obtains the β-action from the α-action by "integrating out" the objects which are on the scale between β and α. (For energy scale formalism these objects are fields with $\alpha \leq$energy$\leq \beta$ and for the length scale the worldlines with $\alpha \leq$length$\leq \beta$.)

Actually, the quadratic part of the action is independent of the scale, so we will consider only the interactions I instead of actions $S = -\frac{g}{2} + I$. Therefore, the Renormalization Group Equation (RGE) that we will deal will really be of the form

$$I_\beta \;=\; W_{\alpha\beta}I_\alpha.$$

4.6.5. *Definition of effective QFT, i.e., properties of actions $S[-]$.* These actions should have the scaled versions of standard properties of actions in physics, for instance the "scaled locality" property. The locality principle in physics roughly requires that physics respects the geometry of space: there are no "spooky" actions at a distance. In the scaled picture this may be broken and one can only require *asymptotic locality* – some notion of approximate locality at the scale α which becomes the standard locality as $\alpha \to \infty$.[(36)]

4.6.6. *Quantization in the framework of effective theories.* By "quantization" we will mean here the passage from a classical action S to an *effective* QFT action $S[-]$, i.e., a family of actions $S[\alpha]$ that describe the quantum system at the scale α.

The effective action $S[-]$ turns out to contain crucial information. This is exemplified in the Costello-Gwilliam construction of the form effective actions.

[36]As usual, one pays for regularization by replacing simple properties of original objects with more complicated approximate versions.

4.6.7. *Non-uniqueness of quantization.* In principle, the classical information represented by the action S may not suffice to describe the quantum system. This is reflected in the multitude of choices of $S[-]$ that our procedure creates – the deformation quantization is performed inductively in powers of $\hbar$ and at each step we get some unknown parameters.

The theoretical part of the additional information that is needed to describe the quantum system usually comes in the form of some symmetry requirement that the quantum system $S[-]$. In good cases this cuts down the number of unknown parameters to a finite small number, and these are then found through experiments.

In fact, canonical quantization procedures exist in physics (say the "minimal subtraction scheme"), but one needs additional reasoning to argue that in a given situation such procedure leads to the description of the physical system. The mathematical formalism we present is even less explicit since our quantization procedure appears to rely on the axiom of choice (a choice of a complement to a vector subspace).

The upshot is that for mathematicians the quantization procedure is just an abstract theorem on existence and classification of quantum systems with the given classical limit S.

4.6.8. *Quantization procedure as renormalization of Feynman amplitudes of graphs.* The point is that the classical action S will be viewed as the closest known approximation of the (possibly idealized) action $S[\infty]$ that describes physics in the presence of objects of arbitrary scale. Then the effective actions should be something like

$$S[\alpha] \;=\; W_{\infty,\alpha}S[\infty] \;=\; W_{\infty,\alpha}S \;\stackrel{\text{def}}{=}\; \lim_{\beta\to\infty} W_{\beta,\alpha}S.$$

This limit does not exist and therefore has to be modified, we say *renormalized*.

The operators $W_{\varepsilon L}$ are intuitively certain Feynman integral and we actually define them as certain sums of Feynman amplitudes of graphs. These amplitudes are are themselves integrals which are defined for finite β but become undefined in the limit $\beta \to \infty$. The renormalization procedure systematically cancels the infinities that appear in this limit.

Remark. We will work with the "length" scale, in this setting the notion of sizes gets inverted so the role of $\beta \to \infty$ will be played by $\varepsilon \to 0$.

4.6.9. *Quantization variable $\hbar$.* As usual, this quantization process also introduces an extra variable, the "quantization" variable $\hbar$. It appears here in quantum actions $S[\alpha]$ because the renormalization flow operators $W_{\varepsilon L}$ are Feynman integrals which contain $\hbar$ and this forces $\hbar$ to appear in quantum actions $S[\alpha]$.

Question. (Franz) Can one explain the appearance of $\hbar$ from the point of view of the renormalization procedure? (Would the procedure formally work without $\hbar$?)

4.6.10. *Energy scale and length scale.* In 4.3 we have outlined two strategies for approximating divergent Feynman weight integrals with well defined weights $w_P^\gamma(I)$ of propagators P. In the energy scale approach the propagator $P_{\leq\Lambda}$ is obtained by truncation of Q^{-1} to the subspace $\mathcal{E}_{\leq\Lambda}$ of fields of energy $\leq\Lambda$. In the length scale picture the propagator $P_{\varepsilon L}$ is and integral $\int_\varepsilon^L\ dl\ K_l$ of the heat kernel of Q (a truncation of $Q^{-1}=\int_0^\infty\ dl\ K_l$).

We will first derive the formula for the Renormalization Group Flow operators in the more intuitive energy picture. Then we will define these operators in the length formalism by replacing $P_{\leq\Lambda}$ with $P_{\varepsilon L}$.

5. Change of scale ("Renormalization group Flow")

5.1. **Relevant classes of functionals on fields.** The basic manifestation of the Locality Principle in physics is that the action functionals are required to be local in the sense below.

5.1.1. *Functional analysis.* As explained in 4.2.1 we will work in the tensor category $(Nuc,\otimes)$ of complete nuclear topological vector spaces with the *completed projective* tensor product $\otimes$. We denote the dual of a vector space by V^* and the space of continuous liner functionals by $V^\vee$.

5.1.2. *Observables, i.e., functions on the space of fields.* We work on the formal neighborhood of zero in the space $\mathcal{E}$ of functionals. So, the class of functions we use are the formal series

$$\mathcal{O}(\mathcal{E})\ \stackrel{\text{def}}{=}\ \prod_{n\geq 0}\mathcal{O}^n(\mathcal{E})\quad\text{where}\quad\mathcal{O}^n(\mathcal{E})\ \stackrel{\text{def}}{=}\ [(\mathcal{E}^{\otimes})^\vee]^{S_n}.$$

So, our functionals have Taylor series expansion into homogeneous components which we denote $I=\sum_0^\infty\ I_k$.

5.1.3. *Local functionals.* A Lagrangian on the space of fields $\mathcal{E}$ is a map $\mathcal{L}:\ \mathcal{E}\to Dens_M$. Any Lagrangian $\mathcal{L}$ gives a functional on the space of fields $S(x)\stackrel{\text{def}}{=}\int_M\ L(x,m),\quad x\in\mathcal{E}$.

A Lagrangian L is said to be *local* if it is a finite sum of products

$$L(x,m)\ =\ \sum_r\ (\prod_i\ D_{i,r}x)(m)$$

for differential operators $D_{i,r}$ on M (with values in densities).[37] In other words, L factors to a function on the jet bundle $JE\to M$ of fields.

A functional $F\in S(\mathcal{E}^*)$ on the space of fields is said to be *local* if (i) it comes from a Lagrangian and (ii) this Lagrangian is local. So, locality for F means that it "localizes

[37]Operators $D:\ \otimes C^{-\infty}(M,V_i)\to C^{-\infty}(M,V)$ of the form $D(x_1\otimes\cdots x_n)=\ \sum_r\ (\prod_{i=1}^n\ D_{i,r}x_i)$ for differential operators $D_{i,r}$ on M are called *polydifferential operators*.

on M" and that the contribution from a point $m \in M$ depends only on the jet $J_m x$ of x at m.

A power series functional $I \in \mathcal{O}(\mathcal{E}) = \widehat{S}(\mathcal{E})$ or $I \in \widehat{S}(\mathcal{E})[[\hbar]]$ is said to be *local* if its homogeneous components are. Local functionals form the subspace $\mathcal{O}_l(\mathcal{E}) \subseteq \mathcal{O}(\mathcal{E})$. We denote $\mathcal{O}_l^n(\mathcal{E}) \stackrel{\text{def}}{=} \mathcal{O}^n(\mathcal{E}) \cap \mathcal{O}_l(\mathcal{E})$.

Remark. Local functionals $\mathcal{O}_l(\mathcal{E}) \subseteq \mathcal{O}(\mathcal{E})$ are not a subalgebra. For instance for $\mathcal{E} = C^\infty(M)$ the square of the local functional $F(x) = \int_M x(m)\, dm$ is $F^2(x) = \int_{(u,v)\in M^2} x(u)y(v)\, du\, dv$.

5.1.4. *The action and interaction functionals.* **(1)** *Classical.* The space $\mathcal{A}ct_c$ of *classical action functionals* is the subspace of the space $\mathcal{O}_l(\mathcal{E})$ consisting of local functionals S which are at least quadratic. We will view it as a sum of subspaces

$$\mathcal{A}ct_c \;=\; \mathcal{O}_l^2(\mathcal{E}) \oplus \mathcal{I}_c \stackrel{\text{def}}{=} \mathcal{O}_l(\mathcal{E})$$

of local quadratic functionals $\mathcal{O}_l^2(\mathcal{E}) \stackrel{\text{def}}{=} \mathcal{O}^2(\mathcal{E}) \cap \mathcal{O}_l(\mathcal{E})$ and the space of *classical interaction functionals* $\mathcal{I}_c \subseteq \mathcal{O}_l(\mathcal{E})$ which consists of local functionals which are at least cubic.

So, any action $S \in \mathcal{A}ct_c$ decomposes into the quadratic part S_2 which we write as $S = -\frac{g}{2}$ and the *interaction* part $I \stackrel{\text{def}}{=} \sum_{i>2} S_i \in \mathcal{I}_c$.

(B) *Quantum.* When passing from classical to quantum actions the quadratic term does not change but interaction acquire formal series in the Planck constant $\hbar$. Therefore, the space of perturbative *quantum actions* is a sum of quadratic local functionals and quantum interactions

$$\mathcal{A}ct_q \stackrel{\text{def}}{=} \mathcal{O}_l^2(\mathcal{E}) \oplus \mathcal{I}_q \;\subseteq \mathcal{O}(\mathcal{E})[[\hbar]]$$

where perturbative *quantum interactions* form the subspace of

$$\mathcal{I}_q \subseteq \mathcal{O}_l(\mathcal{E})[[\hbar]]$$

given by the requirement that the constant term $I|_{\hbar=0}$ in the $\hbar$-expansion is a classical interaction, i.e., it is at least cubic.

(C) *Effective Quantum.* In the framework of effective QFT one is not working with a single quantum action $S = -\frac{g}{2} + I$ but rather with a family of compatible "effective" actions $S[\alpha] = -\frac{g}{2} + I[\alpha],\ 0 < \alpha < \infty$. It turns out that none of these actions has to be local but the family $S[\alpha],\ 0 < \alpha < \infty$, will be required to have certain locality property. So, $I[\alpha]$ is only required to be in the larger subspace $\widetilde{\mathcal{I}}_q \subseteq \mathcal{O}(\mathcal{E})[[\hbar]]$ given by the condition that $I[\alpha]||_{\hbar=0}$ is at least cubic.

5.2. **Scaling flow ("Renormalization Group Flow" or RGF) in energy picture.** Here we derive a formula for the renormalization group operators in the setting of the energy scale. The energy scale provides the framework of spectral theory of the generalized Laplace Q on the space of fields. This allows us to work with well defined Feynman

integrals. At the end we restate the formula in terms of graph expansions, this gives us an abstract notion of RGF operators associated to a choice of a scale.

5.2.1. *The physics justification for scaling the notion of QFT.* Our understanding of physics clearly depends on the scale, for instance the physics on drastically different scales is described by different theories, Moreover, even within one physical theory the scale may be present because in any given experiment we have a bound Λ on energy that can be used. So, it is possible that within a single theory the physics depends on the scale Λ and that it requires slightly different descriptions for different Λ.

5.2.2. *Effective theory via spectral truncation of the operator Q.* In energy picture the spectral analysis of Q gives decompositions $\underline{\mathcal{E}} = \underline{\mathcal{E}}_{\leq\Lambda} \oplus \underline{\mathcal{E}}_{>\Lambda}$ into spaces of fields with specified energies. The projection $\underline{\mathcal{E}} \twoheadrightarrow \underline{\mathcal{E}}_{\leq\Lambda}$ gives the embedding of "observables on scale $\leq \Lambda$" (the measurements, i.e., experiments that can be performed with $\leq \Lambda$ energy)

$$Ob_{\leq\Lambda} \stackrel{\text{def}}{=} \mathcal{O}(\underline{\mathcal{E}}_{\leq\Lambda})$$

into the space $Ob \stackrel{\text{def}}{=} \mathcal{O}(\mathcal{E})$ of all observables.

We will now *assume* that the physics at scale $\leq \Lambda$ is described by an action which we denote $S[\Lambda] \in Ob_{\leq\Lambda}$.[38]

5.2.3. *Correlators on a given scale.* Let $\phi \in \mathrm{Ob}_{\leq\Lambda}$ be an observable on a scale Λ, i.e., ϕ only sees fields of energy $\leq \Lambda$, its expectation $\langle\phi\rangle$ is computed in terms of the action S_Λ on the scale Λ

$$\langle\phi\rangle \;=\; \int_{x\in\mathcal{E}_{\leq\Lambda}} dx \;\; e^{S_\Lambda(x)/\hbar}\, \phi(x)\ .$$

Notice that here our Feynman integral is well defined since it is on a finite dimensional space $\mathcal{E}_{\leq\Lambda}$ (there are finitely many eigenvalues $\lambda \leq \Lambda$!). This is the promised feature of the effective theory that everything is well defined ("no infinities").

5.2.4. *Renormalization Group Equation (RGE).* For two scales $\Lambda' \leq \Lambda$ any Λ'-observable ϕ is also a Λ-observable (formally, the projection $\mathcal{E}_{\leq\Lambda} \twoheadrightarrow \mathcal{E}_{\leq\Lambda'}$ along $\mathcal{E}_{(\Lambda',\Lambda]}$ gives the inclusion $\mathrm{Ob}_{\leq\Lambda'} \subseteq \mathrm{Ob}_{\leq\Lambda}$). This provides two ways to calculate $\langle\phi\rangle$ by using either $S_{\Lambda'}$ or S_Λ. The equality of two formulas of $\phi\rangle$ is easily seen to be equivalent to the following relation between actions $S_{\Lambda'}$ and S_Λ. We call it the *Renormalization Group Equation*.

[38] This is the marriage of the "action principle" (physics is described by an action) with the "effective theory principle" (physics can be observed only on a bounded scale at any given time).

Lemma. For $\Lambda' \leq \Lambda$ and any field $x' \in \mathrm{Ob}_{\leq \Lambda'}$

$$S_{\Lambda'}(x') \;\; = \;\; \hbar \; \log[\int_{y \in \mathcal{E}_{(\Lambda', \Lambda]}} dy \; e^{S_\Lambda(x'+y)/\hbar} \;].$$

Proof. For $\phi \in \mathrm{Ob}_{\leq \Lambda'}$,

$$\int_{x' \in \mathcal{E}_{\leq \Lambda'}} dx' \;\; e^{S_{\Lambda'}(x')/\hbar} \; \phi(x') \;\; = \;\; \langle \phi(x) \rangle \;\; = \;\; \int_{x \in \mathcal{E}_{\leq \Lambda}} dx \;\; e^{S_\Lambda(x)/\hbar} \; \phi(x)$$

can be rewritten in terms of $x = x' + y$ as

$$= \int_{x' \in \mathcal{E}_{\leq \Lambda'}} dx' \;\; \int_{y \in \mathcal{E}_{(\Lambda', \Lambda]}} dy \;\; e^{S_\Lambda(x'+y)/\hbar} \; \phi(x'+y).$$

Here, $\phi(x'+y) = \phi(x')$ since $\phi \in \mathrm{Ob}_{\leq \Lambda'}$, when viewed as a function on $\mathrm{Ob}_{\leq \Lambda}$ is invariant under translations by $\mathrm{Ob}_{(\Lambda', \Lambda]}$. So, the integral is

$$= \int_{x \in \mathcal{E}_{\leq \Lambda'}} dx' \; \phi(x') \; \int_{y \in \mathcal{E}_{(\Lambda', \Lambda]}} dy \; e^{S_\Lambda(x'+y)/\hbar} \; .$$

This gives

$$e^{S_{\Lambda'}(x'/\hbar)} \;\; = \;\; \int_{y \in \mathcal{E}_{(\Lambda', \Lambda]}} dy \; e^{S_\Lambda(x'+y)/\hbar} \; .$$

Remark. The RGE says that for $\Lambda > \lambda'$ $S_{\Lambda'}$ is obtained from S_Λ by "allowing the field x' on the scale $\leq \Lambda'$ to fluctuate on the scale $(\Lambda', \Lambda]$", i.e., by averaging over fluctuations on scale $(\Lambda', \Lambda]$. So, one is just integrating out the objects on the intermediate scale $(\Lambda', \Lambda]$. The same words describe the transition on the level of interactions rather then actions:

5.2.5. *RGE for interactions.* The quadratic part $-\frac{g}{2}$ of the action is expected to be the same at all scales (in particular the same as in the classical action!). Therefore,

$$S_\Lambda \;\; = \;\; -\frac{g}{2} + I_\Lambda$$

and we will only be interested in the evolution of the interaction party $I_\Lambda \in \mathcal{I}_q$ with the scale. The corresponding form of RGE is:

Corollary. For $\Lambda' \leq \Lambda$ and any field $x' \in \mathrm{Ob}_{\leq \Lambda'}$

$$I_{\Lambda'}(x') \;\; = \;\; \hbar \; \log[\int_{y \in \mathcal{E}_{(\Lambda', \Lambda]}} dy \; e^{-\frac{1}{\hbar}[-\frac{g(x')}{2} + I_\Lambda(x'+y)]} \;].$$

Proof. We know that

$$e^{-\frac{1}{\hbar}[I_{\Lambda'}(x')]} \;\; = \;\; e^{\frac{g(x')}{2} + S_{\Lambda'}(x'/\hbar)} \;\; = \;\; \int_{y \in \mathcal{E}_{(\Lambda', \Lambda]}} dy \; e^{\frac{g(x')}{2} + S_\Lambda(x'+y)/\hbar} \; .$$

Since

$$\frac{g(x')}{2} + S_\Lambda(x'+y) \;=\; \frac{g(x')}{2} - \frac{g(x'+y)}{2} + I_\Lambda(x'+y) \;=\; -g(x',y) - \frac{g(y)}{2} + I(x'+y),$$

we are done by orthogonality of $x' \in \mathcal{E}_{\leq \Lambda'}$ and $y \in \mathcal{E}_{(\Lambda',\Lambda]}$.

5.2.6. *Renormalization group flow W.* The RGE says that the transition between scales is given by the operator $W_{\Lambda',\Lambda}$ defined on the space $\mathcal{I}_q$ of quantum interactions by

$$(W_{\Lambda',\Lambda} I)(x') \;\stackrel{\text{def}}{=}\; \hbar \, \log \Big[\int_{y \in \mathcal{E}_{(\Lambda',\Lambda]}} dy \; e^{\frac{1}{\hbar}[-\frac{g(x')}{2} + I(x'+y)]} \Big].$$

These are the *Renormalization Group Flow* operators on $\mathcal{I}_q$ (in the energy scale formalism).

The RGF operators are the basic objects in the effective QFT formalism. The procedure of quantization of classical theories used below is just the extension of the construction $W_{\Lambda',\Lambda}$ to $L = \infty$, i.e., to the setting where fields of all scales are allowed.

Remarks. (0) One should keep in mind that by our definitions I_Λ is a formal series in fields so it is only defined on infinitesimal fields. So, the formulas above are meaningful (and correct) only for infinitesimal elements x' of $\mathcal{E}_{\leq \Lambda'}$, but this still gives a well defined operator on interaction functions which are defined only infinitesimally on $\mathcal{E}$.

(1) On the other hand, once we remember that we are working on formal neighborhoods of zero, these RGF operators are well defined since we are integrating over a finite dimensional vector space $\mathcal{E}_{(\Lambda',\Lambda]}$.

5.2.7. *Renormalization graphs.* Our combinatorial setting is the class of graphs $\mathcal{TG}^\bullet$, the *graded graphs with tails*. These are pairs $\gamma = (\Gamma, g)$ of a graph with tails, i.e., a diagram of finite sets

$$\Gamma \;=\; (P_\Gamma \stackrel{\sigma}{\leftarrow} P_\Gamma \stackrel{\pi}{\rightarrow} V_\Gamma)$$

where σ is an involution (see 3.6.1), and a "grading" $g : V_\gamma \stackrel{\text{def}}{=} V_\Gamma \longrightarrow \mathbb{N}$ which is called genus. The genus of the graph γ is the integer

$$g_\gamma \stackrel{\text{def}}{=} b_1(\gamma) + \sum_{v \in V_\gamma} g_v.$$

The subclass $\mathcal{RG}$ of "renormalization graphs" is the class $c\mathcal{TG}^\bullet$ of connected graded graphs with tails.

We say that γ is *stable* if

- vertices of genus 0 are at lest trivalent and
- if vertices of genus 1 are at least one valent.

The subclass $\mathcal{RG}$ of "*renormalization graphs*" is the class $cs\mathcal{TG}^\bullet$ of connected stable graded graphs with tails.

5.2.8. *Graph expansion of RGF operators.* Since the Feynman integral in the definition of $W_{\Lambda',\Lambda}I$ is finite dimensional it has a valid graph expansion. To write it down, notice that the integral uses the restriction $g|_{\mathcal{E}_{(\Lambda',\Lambda]}}$ of the metric g on $\mathcal{E}$ to a metric on $\mathcal{E}_{(\Lambda',\Lambda]}$. This restriction is represented by the restriction of $Q|_{\mathcal{E}_{(\Lambda',\Lambda]}}$ of Q. The propagator for $g|_{\mathcal{E}_{(\Lambda',\Lambda]}}$ is therefore $(Q|_{\mathcal{E}_{(\Lambda',\Lambda]}})^{-1} = Q^{-1}|_{\mathcal{E}_{(\Lambda',\Lambda]}}$. We denote by $P_{(\Lambda',\Lambda]}$ the operator on $\mathcal{E}$ which is the extension of $Q^{-1}|_{\mathcal{E}_{(\Lambda',\Lambda]}}$ by 0 on $\mathcal{E}^{\perp}_{(\Lambda',\Lambda]}$, i.e., the $(\Lambda',\Lambda]$-spectral truncation of Q^{-1}.

The expansion is over the moduli $\mathcal{RG}$ of renormalization graphs, introduced above.

Lemma. The RGF operators for energy scale have an exact[39] Feynman expansion

$$(W_{\Lambda',\Lambda}I)(x') \;=\; \int_{\gamma\in\mathcal{RG}} \hbar^{g_\gamma}\; w^{\gamma}_{P_{(\Lambda',\Lambda]}}(I).$$

Proof. In the expansion of

$$\int_{y\in\mathcal{E}_{(\Lambda',\Lambda]}} dy\; e^{\frac{1}{\hbar}[-\frac{g(x')}{2}+I(x'+y)]}$$

the graphs should be graded as in 3.5.3 because this Feynman integral contains the Planck constant $\hbar$. However, the formula in 3.5.3

$$\int_{\gamma\in\mathcal{G}^\bullet} \hbar^{-\chi_\gamma+\sum_{v\in V_\gamma}\; g_v}\; w^{\gamma}_{Q^{-1}|_{\mathcal{E}_{(\Lambda',\Lambda]}}}(I)$$

must be modified in one aspect – the infinitesimal additive shift $y\mapsto y+x'$ in the integral replaces the class $\mathcal{G}$ of graphs with the class $\mathcal{TG}$ of graphs with tails. So, the relevant class $\mathcal{TG}^\bullet$ is that of graded graphs with tails. Finally, to calculate the logarithm we just restrict to the connected graphs $c\mathcal{TG}^\bullet$. So,

$$\log\Big[\;\int_{y\in\mathcal{E}_{(\Lambda',\Lambda]}} dy\; e^{\frac{1}{\hbar}[-\frac{g(x')}{2}+I(x'+y)]}\;\Big] \;=\; \int_{\gamma\in c\mathcal{TG}^\bullet} \hbar^{-\chi_\gamma+\sum_{v\in V_\gamma}\; g_v}\; w^{\gamma}_{Q^{-1}|_{\mathcal{E}_{(\Lambda',\Lambda]}}}(I).$$

Once we multiply this with $\hbar$, the power of $\hbar$ that appears with the graph γ equals g_γ since the power is $\sum_{v\in V_\gamma}\; g_v$ plus $1-\chi_\gamma = 1-(b_0(\gamma)-b_1(\gamma)) = b_1(\gamma)$ (since the graph is connected).

Finally, for $I\in \mathrm{Ob}_{\leq\Lambda}$, the contraction $w^{\gamma}_{Q^{-1}|_{\mathcal{E}_{(\Lambda',\Lambda]}}}(I)$ does not change if we replace $Q^{-1}|_{\mathcal{E}_{(\Lambda',\Lambda]}}$ by its extension $P_{\mathcal{E}_{(\Lambda',\Lambda]}}$ to $\mathcal{E}$ by zero on $\mathcal{E}^{\perp}_{(\Lambda',\Lambda]}$.

! Should explain the stability condition.

[39]Here, "exact" is used to emphasize that both sides are well defined and we have an equality of well defined quantities.

5.3. **Scaling flow ("Renormalization Group Flow") in length picture.** For the length scale we will define the RGF operators by following the abstract notion of RGF that we found by considering the energy scale. When we considered the energy scale then the scale appeared on the level of fields and we started by truncating the fields according to the scale. However, the end result was that the essential ingredient is the notion of truncating the ideal propagator Q^{-1} to actual propagators and this can also be done for the length scale (5.3.1).

The notion of the *length scale* will here enter in a formal way and only later it will be given a geometric interpretation (which is geometrically appealing but not manifestly physical). The length will be defined as the parameter l in the formula

$$Q^{-1} = \int_0^\infty dl\ e^{-lQ}.$$

The formula obviously provides truncations of Q^{-1} with respect to the parameter l,

$$P_{\varepsilon L} \stackrel{\text{def}}{=} \int_\varepsilon^L dl\ e^{-lQ}.$$

The operators $e^{-lQ},\ l \geq 0$, have integral kernels K_l and the family K is known as the *heat kernel* for the generalized Laplace operator Q. This provides a tool for understanding the singularity of Q^{-1} in terms of the asymptotics of heat kernels.

According to the scheme 5.3.1, these l-truncations of Q^{-1} provide a notion of RGF operators for the length scale l by the graph expansion

$$(W_{\varepsilon,L}I)(x') \stackrel{\text{def}}{=} \int_{\gamma\in\mathcal{RG}} \hbar^{g_\gamma}\ w^\gamma_{\varepsilon L}(I)$$

for $w^\gamma_{\varepsilon L} \stackrel{\text{def}}{=} w^\gamma_{P_{\varepsilon L}}$.

The operators obtained by this formal definition actually have a geometric interpretation as a special case of the "particle" interpretation of Feynman integrals in the heat kernel formalism from 4.5.

5.3.1. *The notion of RGF operators for scale $\mathcal{S}$.* By a *scale* $\mathcal{S}$ for a free theory (M, E, Q) we will mean any truncation of the ideal propagator Q^{-1} to a family of propagators $P_{\alpha\beta} \in S^2\mathcal{E}$ for $(\alpha, \beta) \in \mathbb{R}^{>0 2}$ such that

- (i) in some sense
$$\lim_{\alpha\to 0,\ \beta\to\infty} P_{\alpha\beta} = Q^{-1}.$$
- $P_{\alpha\beta} + P_{\beta\gamma} = P_{\alpha\gamma}$.
- All $P_{\alpha\beta}$ commute,

We think of $P_{\alpha\beta}$ as the truncation of Q^{-1} to the $\mathcal{S}$-scales $(\alpha, \beta]$ intermediate between α and β.

To a scale $\mathcal{S}$ we associate the $\mathcal{S}$-RGF $W^{\mathcal{S}}$ consisting of the operators on $\mathcal{I}_q$

$$(W^{\mathcal{S}}_{\alpha,\beta} I)(x') \stackrel{\text{def}}{=} \int_{\gamma\in\mathcal{RG}} \hbar^{g_\gamma}\, w^{\gamma}_{P^{\mathcal{S}}_{\alpha,\beta}}(I) \;=\; \hbar \log[\underline{e^{\frac{1}{2}P^{\mathcal{S}}_{\alpha,\beta}}}(e^{?})].$$

For $\beta > \alpha$ we think of this as integrating out the degrees of freedom that on the $\mathcal{S}$-scale lie between α and β.

We define the scale $\mathcal{S}$ RGE for a family of quantum interactions $I_\alpha = I[\alpha] \in \mathcal{I}_q,\ 0 < \alpha < \infty$, as

$$I_\beta \;=\; W_{\alpha\beta} I_\alpha, \qquad \alpha, \beta > 0.$$

Remark. In the present terminology the energy scale consists of spectral truncations $P_{\Lambda',\Lambda}$ of Q^{-1} relative to the Q-spectral subspaces $\mathcal{E}_{(\Lambda',\Lambda]}$. In this case the Q-spectral decomposition of fields provides extra data – a notion of the theory I_Λ truncated to the scale $\leq \Lambda$. Then the requirement of consistency for these theories (the correlators for $\leq \Lambda'$ observables can be calculated in any truncation $\leq \Lambda$ with $\Lambda \geq \Lambda'$) forces the transition between between truncated theories $I_{\Lambda'}, I_\Lambda$ to be given by operators $W_{\Lambda'\Lambda}$ that are given by the above formula (lemma 5.2.8).

The abstract form of the formula in lemma 5.2.8) is the source of the above definition of a general notion of a scale and of the associated RGF and RGE.

5.3.2. We define the length-RGE for a family of quantum interactions $I_L = I[L] \in \mathcal{I}_q,\ 0 < L < \infty$, as

$$S_L \;=\; W_{\varepsilon L}\, S_\varepsilon.$$

5.4. **The scaling connection.** Peter Dalakov and Aaron Gerding suggested that one should also view the length-RGF as a connection. Recall that while the energy RGF operators $W_{\Lambda',\Lambda}$ are not continuous in the scale (they jump at eigenvalues of Q), the length RGF operators $W_{\varepsilon L}$ are smooth in the scale.

This connection is the simplest in the space of exponentials $J = e^{I/\hbar}$ where it becomes linear. However, I is a more fundamental quantity than J – for one thing we understand the locality property in terms of I but not in terms of J.

5.4.1. *The "scaling" connection.* $W_{\varepsilon L}$ is the parallel transport on the trivial vector bundle $V = (0,\infty)\times\widetilde{\mathcal{I}}_q$ over $(0,\infty)$. The corresponding connection is $\nabla = d - W_L$ where operators W_L on $\widetilde{\mathcal{I}}_q$ are

$$W_L \stackrel{\text{def}}{=} (\frac{d}{dL} W_{\varepsilon L})|_{\varepsilon = L}.$$

We also consider this connection in terms of the variable $J = e^{I/\hbar}$, then the notation is $\mathcal{W}_{\varepsilon L}$ and $\nabla = d - \mathcal{W}_L$.

Lemma. (a) The connection ∇ is given in terms of variables I and J by operators

$$W_L(I) \;=\; \frac{1}{e^{I/\hbar}}\;\frac{1}{2}\hbar\underline{K_L}e^{I/\hbar} \quad\text{and}\quad \mathcal{W}_L J \;=\; \frac{1}{2}\hbar\underline{K_L}\;J.$$

(b) Let $\mathcal{RG}1\subseteq\mathcal{TG}$ be the submoduli of renormalization graphs with one edge, then

$$W_L \;=\; \int_{\gamma\in\mathcal{RG}1} w^\gamma_{K_L}(I).$$

Proof. (a) Since $W_{\varepsilon L}=\hbar\log[\underline{e^{\frac{1}{2}P_{\varepsilon L}}}\;e^{I/\hbar}]$,

$$\frac{d}{dL}W_{\varepsilon L}|_{\varepsilon=0} \;=\; \hbar(\underline{e^{\frac{1}{2}P_{\varepsilon L}}}\;e^{I/\hbar})^{-1}\;\frac{1}{2}\underline{K_L}e^{I/\hbar}|_{\varepsilon=L} \;=\; \frac{1}{e^{I/\hbar}}\;\frac{1}{2}\hbar\underline{K_L}e^{I/\hbar}.$$

In terms of $J=e^{I/\hbar}$, the parallel transport and the connection are simply

$$\mathcal{W}_{\varepsilon L}J \;=\; \underline{e^{\frac{1}{2}\hbar P_{\varepsilon L}}}\;J \quad\text{and}\quad \mathcal{W}_L J \;=\; \frac{1}{2}\hbar\underline{K_L}\;J.$$

(b) Recall that $w^\gamma_P(I)=\langle P_\gamma, I_\gamma\rangle_\gamma$ is a γ-contraction of two tensors which have been localized (placed) on γ. We have $P_\gamma=P^{E_\gamma}$ with one propagator P placed on each edge $e\in E_\gamma$ and $I_\gamma=\otimes_{v\in V_|ga}\;I_{k_v}$ where at each vertex $v\in V_\gamma$ of valency k_v we place I_{k_v}. Then one contracts along the prongs (half-edges) of γ. This produces $w^\gamma_P(I)\in S(\mathcal{E}^*)$ of the degree t_γ, the number of tails of γ.

Now $w^\gamma_{\varepsilon L}(I)$ is a function of L through its propagator $P=P_{\varepsilon L}$ and $\frac{d}{dL}P_{\varepsilon L}=K_L$. Therefore, applying $\frac{d}{dL}$ to $w^\gamma_{P_{\varepsilon L}}(I)$ we get a sum over edges e where the e-summand is obtained by differentiating $P_{\varepsilon L}$ placed at the edge e, i.e., at the edge e we replace $P_{\varepsilon L}$ by K_L.

When we plug in $\varepsilon=L$ at all edges $e'\neq e$, $P_{\varepsilon L}$ becomes $P_{L,L}=0$. Therefore, if γ has more then one edge all summands become zero, and if γ has a single edge then $\frac{d}{dL}w^\gamma_{\varepsilon L}=w^\gamma_{K_L}(I)$.

Corollary. The partition of $\mathcal{RG}1$ into graphs $\mathcal{RG}1^{loop}$ where the single edge is a loop and $\mathcal{RG}1^{edge}$ where the edge is not a loop, decomposes the operator W_L as $W_L^{edge}+W_L^{loop}$ and

$$W_L^{loop} \;=\; \hbar\underline{K_L}$$

while

Here,

$$\hbar^i S^k(\mathcal{E}^*)\xrightarrow{W_L^1}\hbar^{i+1}S^{k-2}(\mathcal{E}^*) \quad\text{and}\quad \hbar^i S^{k'}(\mathcal{E}^*)\otimes\hbar^{i'}S^{k''}(\mathcal{E}^*)\xrightarrow{W_L^0}\hbar^{i'+i''}S^{k'+k''-2}(\mathcal{E}^*).$$

Proof. Any graph γ in $\mathcal{RG}1^1$ has a single vertex c, we place I_{k_c} at c and contract $I_\gamma=I_{k_c}$ with K_L along e. The effect of the operator

$$W_L^1(I)=\int_{\gamma\in\mathcal{RG}^1}\hbar^{g_\gamma}w^\gamma_{K_L}(I)$$

is just the differentiation of I by $\underline{K_L}$.

Here, $g_\gamma = b_1(\gamma) + g_c = g_c + 1$.

If $e \in \mathcal{RG}1^0$ then e connects two different vertices a, b and we place I_{k_a} at a and I_{k_b} at b and contract $I_\gamma = I_{k_a} \otimes I_{k_b}$ with K_L along e. Here, $g_\gamma = g_a + g_b$.

"Poisson" structure

$$[\{I', I''\}_L]_{ik} \stackrel{\text{def}}{=} \sum i' + i'' = i,\ k' + k'' = k\ \langle I_{i'k'} I_{l''k''}, K_L \rangle$$

where the summand means all K_L-contractions of $I_{i'k'} \otimes I_{l''k''}$ along pairs of prongs, one in each flower. .....

BV-Laplacian?

Remark. Connection ∇ is quadratic in terms of the variable I which is 'local" in M. The singularity at $\varepsilon = 0$ appears only in the linear part W_L^1. In terms of J the connection is linear, however J is only "exponentially local" in M¿

Question. Is this the root of the 'exponential locality" of factorization algebras?

5.4.2. *The universal solution.* This is the operator of parallel transport to infinity

$$W_L \stackrel{\text{def}}{=} W_{L\infty} = \underline{e^{-P_{L\infty}}}.$$

Question. Does $\underline{e^{-P_{L\infty}}}$ have a kernel? Is $e^{-P_{L\infty}}$ its "kernel"?

Proof.

$$\frac{d}{dL} e^{-P_{L\infty}} = e^{-P_{L\infty}} \frac{d}{dL}(-P_{L\infty}) = K_L e^{-P_{L\infty}} \frac{d}{dL}.$$

5.5. **The definition of Effective Quantum Field Theories.**

5.5.1. *The* free *data* (M, E, g_E, Q). They consist of a vector bundle (g on a manifold M, which carries a generalized Laplace operator Q and a metric g_E.

The symbol of Q gives a metric g_M on M ($\sigma(Q) = g_M^*$). This metric gives a measure dm on M. Then g_E and dm produce a metric $g = g_\mathcal{E}$ on the space $\mathcal{E} = C^\infty(M, E)$ of sections of E.

5.5.2. *Wilson's definition of QFT in terms of effective actions for the energy scale.* Wilson defined an *effective QFT* for a free theory (E, Q, g_E), as a family of "effective actions" $S[\Lambda] = S_\Lambda$ indexed by energy scales $0 < \Lambda < \infty$ and satisfying the following principles:

(1) Each S_Λ is a power series in $\hbar$ and in the fields: $S_\Lambda \in \widehat{S}(\mathcal{E}^*)[[\hbar]]$.
The quadratic part of $S[\Lambda]$ does not depend on the scale and it comes from the generalized Laplace operator:

$$S[\lambda] \;=\; -\frac{g}{2} + I_\Lambda,$$

where the *interaction part* $I_\Lambda \in \widehat{S}(\mathcal{E}^*)[[\hbar]]$ is at least cubic in fields.
(2) $S[\Lambda]$ is a function on $\mathcal{E}_{\leq\Lambda}\subseteq\mathcal{E}$ the fields of energy $\leq \Lambda$, i.e., S_Λ factors through the projection $\mathcal{E}\twoheadrightarrow\mathcal{E}_{\leq\Lambda}$.
(3) [*Renormalization Group Equation.*] For $\Lambda' < \Lambda$, $S[\Lambda']$ is determined from $S[\Lambda]$ by the RGE:

$$I'_\Lambda \;=\; W_{\Lambda',\Lambda} I_\Lambda.$$

(4) [*Locality.*] The family S_Λ satisfies certain Λ-notion of locality axiom formulated below in 5.5.6.

5.5.3. *Principle of locality.* This is a requirement of compatibility of physics with the space M on which we observe it. The intuitive version says that

Interactions between fundamental particles only happen at points.

In other words, particles only interact when they are at the same point of space. One can state it as

NO *spooky action at a distance.*

So, if particles interact at a distance this is through the mediation of other particles between them.

In terms of the action, locality principle appears as the requirement that the action is a *local functional* on fields. This means that it is the integral of local contributions

$$S(x) \;=\; \int_{m\in M} L(x,m)$$

and the local contribution $L(x,m)$ (called *Lagrangian*) is local in the sense that it is a function on the jet bundle JE of fields, i.e., it is a finite sum of products $(\prod_i D_i x)(m)$ for differential operators D_i. yyy

5.5.4. *Asymptotic locality.* The effective action formalism is not directly compatible with locality – the locality principle holds only when we include all scales. At any given range of scales locality only holds approximately and this approximation gets better as we extend the range. A precise formulation is that locality holds *asymptotically* (the definition of asymptotic expansions is in 5.7 below).

We say that a solution $I[-]$ of the scale $\mathcal{S}$ RGE is *asymptotically local* if each of its components in the expansion

$$I[\alpha] \;=\; \sum_{i,k\geq 0} \hbar^i \frac{I_{ik}[\alpha]}{k!}$$

has an asymptotic expansion for *large* α (meaning "as all scales get included"), as a linear combination of local functionals $\Theta_{ik,r}$, with coefficients which are functions $f_{ik,r}C^\infty[0,\infty)$ of the scale α :

$$I_{ik}[\alpha](x) \stackrel{AE}{=} \sum_{r=0}^{\infty} f_{ik,r}(\alpha)\; \Theta_{ik,r}(x).$$

Remark. For energy scale this formalism is not compatible with the RGF action – if S_Λ is close to local then the RGE $S_{\Lambda'} = W_{\alpha\beta}S_\Lambda$ forces other $S_{\Lambda'}$ to be "completely nonlocal". However, the asymptotic locality requirement is compatible with the length scale. ("Length" is more geometric, i.e., local, while energy is more a matter of global harmonic analysis.)

In this aspect the energy scale is more cumbersome, for instance its renormalization schemes are more complex precisely because it is difficult to control preservation of locality. We will use the length scale picture which makes renormalization transparent.

5.5.5. *The definition of effective QFT in terms of the length scale.* An *effective QFT* for a free theory (E, Q, g_E), is a family of "effective" quantum interactions $I[L] = I_L \in \mathcal{I}_q$ indexed by the length scales $0 < L < \infty$ and satisfying the following principles:

(1) [*The definition of the space $\mathcal{I}_q$ of quantum interactions.*] Each I_L is a power series in $\hbar$ and in the fields: $I_L \in \widehat{S}(\mathcal{E}^*)[[\hbar]]$ and the specialization to $\hbar = 0$ is a classical interaction, i.e., it is at least cubic in the fields (and components are local functionals).

(2) [*Renormalization Group Equation.*] For any $0 < \varepsilon, L < \infty$,

$$I_{\mathbb{L}} \;=\; W_{\varepsilon L} I_\varepsilon.$$

(3) [*Locality.*] The family $I[L]$ is asymptotically local as $L \to 0$, i.e., each of its components in the expansion $I[L] \;=\; \sum_{i,k\geq 0}\; \hbar^i \frac{I_{ik}[L]}{k!}$ has an asymptotic expansion for *small* L[40] as a linear combination of some local functionals $\Theta_{ik,r}$, with coefficients which are functions $f_{ik,r}C^\infty[0,\infty)$ of the scale L :

$$I_{ik}[L](x) \stackrel{AE}{=} \sum_{r=0}^{\infty} f_{ik,r}(L)\; \Theta_{ik,r}(x).$$

[40]For the length scale all scales get included as $L \to 0$.

5.5.6. *Locality in energy picture.* To define it we use the translation between energy and length scales indicated in 4.3.6. So, the meaning of locality axiom in 5.5.2 is that the family S_Λ of energy scales actions becomes asymptotically local when it is translated into a family of length scale effective actions. We notice that the translation procedure is compatible with RGE:

Lemma. The translation between energy and length scales induces a bijection of solutions of the length-RGE and solutions of the energy-RGE which are of "moderate growth".

5.6. **Strong unipotency of the family of operators $W_P,\ P \in S^2\mathcal{E}$.** We order $\mathbb{N}^2$ lexicographically by "$\hbar$ before $\mathcal{E}$", i.e., $(i,k) < (i',k')$ if $i < i'$ or $i = i'$ and $k < k'$. For $\Theta \subseteq \mathbb{N}^2$ denote $\mathcal{I}_{q,\Theta} \stackrel{\text{def}}{=} \prod_{(i,k)\in\Theta} \hbar^i \mathcal{I}_q^{(k)}$ and let $I \mapsto I_\Theta \stackrel{\text{def}}{=} \sum_{(i,k)\in\Theta} \hbar^i I_{ik}$ be the projector $\mathcal{I}_q \to \mathcal{I}_{q,\Theta}$.

similarly, we define the Θ-component of the operator $W = W_{\varepsilon L}$ by $W_\Theta : \mathcal{I}_q \to \mathcal{I}_{q,\Theta}$ with $W_\Theta I \stackrel{\text{def}}{=} (WI)_\Theta$.

Lemma. Let P be a propagator and $\alpha \in \mathbb{N}^2$.

(a) The α-component of $W_P(I)$ only depends on $I_{\leq\alpha}$:

$$[W_P(I)]_\alpha \ = \ [W_P(I_{\leq\alpha})]_\alpha.$$

(b) More precisely,

$$[W_P(I)]_\alpha \ - \ W_{P,\alpha}(I_{<\alpha}) \ = \ I_\alpha.$$

(c) W_P is an automorphism of the $\mathcal{I}_{q,\alpha}$-torsor $\mathcal{I}_{q,\leq\alpha} \twoheadrightarrow \mathcal{I}_{q,<\alpha}$, i.e., for $I \in \mathcal{I}_{q,\leq\alpha}$ and $J \in \mathcal{I}_{q,\alpha}$

$$[W_P(I+J)]_{\leq\alpha} \ = \ [W_P I]_{\leq\alpha} \ + \ J.$$

Sublemma. ($\star$) If $I_{r,s}$ appears in the product $I_\gamma = \prod_{v\in V_\gamma} I_{g_v,k_v} \in S(\mathcal{E}^*)$ for a graph $\gamma \in \mathcal{RG}$ (in other words, if there is a vertex v of γ with genus r and valency s), then in $\mathbb{N}^2$ we have

$$(r,s) \ \leq \ (g_\gamma, t_\gamma).$$

Moreover, equality happens for precisely one graph γ, the "(r,s)-*star*" graph $\star_{r,s} \in \mathcal{RG}$ which consists of one internal vertex v with s tails and of genus $g_v = r$.

Proof of the sublemma ($\star$). Suppose that $I_{r,s}$ appears at the vertex v. Clearly, $r = g_v \leq g_\gamma$. Now, if $g_\gamma > r$ then $(g_\gamma, k_\gamma) > (r,s)$, so we only need to consider the case $r = g_v$ and check that $s \leq k_\gamma$ and that the equality $s = k_\gamma$ holds iff $\gamma \cong \star_{r,s}$.

However, $r = g_v$ implies $g_v = g_\gamma$ and this tells us that

- (i) $b_1(\gamma) = 0$, i.e., γ is a tree and
- (ii) vertices $u \neq v$ have genus 0, so stability implies that their valency k_u is ≥ 3.

Now we will observe that the external valency of γ is

$$t_\gamma \;=\; k_v \;+\; \sum_{u\in V_\gamma - \{v\}} k_u - 2 \;=\; k_\gamma + 1 - 2|V_\gamma|.$$

This will imply the remaining claims of the sublemma ($\star$) since $k_u \geq 3$ gives

$$t_\gamma - k_v = \sum_{u\in V_\gamma - \{v\}} k_u - 2 \;\geq\; |V_\gamma - \{v\}| \geq 0,$$

so $t_\gamma - k_v \geq 0$ and equality is equivalent to v being the only vertex.

The above formula is clear when presented as a picture but longer when described in words. For this we grow the graph γ from the root v. Let γ_1 be the subgraph of γ that consists of the vertex v and all prongs from v, clearly $t_{\gamma_1} = k_v$. Now we obtain γ_2 from γ_1 by adding the vertices of γ that appear at the ends of tails of γ_1 and the prongs that stem from these vertices. Each of the new vertices u contributes $k_u - 2$ to $t_{\gamma_2} - t_{\gamma_1}$. The -2 comes from killing the prong in γ_1 that goes from γ_1 to u and also the one from u towards γ_1. Etc.

Proof of the lemma. Let $\alpha = (i,k) \in \mathbb{N}^2$. Recall that

$$(W_P I)_\alpha = \int_{\mathcal{RG}_\alpha} \hbar^{g_\gamma} w_P^\gamma(I) \quad \text{and} \quad w_P^\gamma(I) = \langle P_\gamma, I_\gamma \rangle$$

$\mathcal{RG}_\alpha \subseteq \mathcal{RG}$ are the graphs γ with $(g_\gamma, t_\gamma) = \alpha$ and the tensor I_γ is obtained by putting at each internal vertex v the tensor I_{g_v,k_v}.

Now (a) follows from the sublemma ($\star$) – if $I_{r,s}$ contributes to the α-component of $W_P I$ then this happens through some $\gamma \in \mathcal{RG}_\alpha$ and then the sublemma says that $(r,s) \leq \alpha$.

(b) The difference

$$[W_P(I)]_\alpha - [W_P(I_{<\alpha})]_\alpha \;=\; [W_P(I_{\leq\alpha})]_\alpha - [W_P(I_{<\alpha})]_\alpha$$

is given by the terms $\hbar^i w_P^\gamma(I_{\leq\alpha})$ for graphs $\gamma \in \mathcal{RG}_\alpha$ such that $(I_{\leq\alpha})_\gamma$ features I_α. The sublemma ($\star$) says that there is just one such graph $\gamma = \star_\alpha$. For this graph, $P_\gamma = 1$ since γ has no edges, hence $w_P^\gamma(I) = I_\gamma$. Also, since γ has just one vertex at which I_α is positioned we have $I_\gamma = I_\alpha$.

(c) follows from (a) and (b),

$$[W_P(I+J)]_{\leq\alpha} \;=\; [W_P(I+J)_{<\alpha}]_{\leq\alpha} + (I+J)_\alpha \;=\; [W_P(I_{<\alpha})]_{\leq\alpha} + I_\alpha + J \;=\; [W_P(I)]_{\leq\alpha} + J.$$

Corollary. For any propagator P, the graph operator W_P preserves the descending filtration of the trivial vector bundle $V = \mathcal{I}_q$ by the trivial vector subbundles $\mathcal{I}_{q,\geq\alpha},\ \alpha \in \mathbb{N}^2$. Moreover, W_P is identity on $Gr(V)$.

Proof. W_P preserves $\mathcal{I}_{q,\geq\alpha}$ since part (a) of the lemma says that for $I \in \mathcal{I}_{q,\geq\alpha}$ and $\beta < \alpha$ we have $(W_P I)_\beta = [W_P\ I_{\leq\alpha}]_\beta = [W_P\ 0]_\beta = 0$. Now, part (b) of the lemma implies that W_P is identity of $Gr_\alpha(V)$.

Remark. In the case when $P = P_{\varepsilon L}$ for the length RGF this has been deduced by calculating the corresponding scaling connection in ....

Question. What is the group $\boldsymbol{W}$ generated by all operators W_P on $\mathcal{I}_q$?

Appendices

5.7. **Appendix A. Asymptotic expansions.** We consider asymptotic expansions (AE) of a function at an endpoint of the interval where function is defined. We will choose the interval to be $(0, \infty)$ with coordinate ε and consider the AE at 0.

The *Asymptotic Expansion* at $\varepsilon = 0$ of a function $\sigma(\varepsilon)$ on $(0, \infty)$ is a sequence of functions $\sigma_n(\varepsilon)$ such that

- For sufficiently large n, σ_n approximates σ for ε near 0 in the following quantitative sense: the error is controlled by ε^{d_n} for some $d_n \geq 0$:
$$\lim_{\varepsilon\to 0} \frac{\sigma(\varepsilon) - \sigma_n(\varepsilon)}{\varepsilon^{d_r}} = 0.$$
- The approximations improve for larger n in the sense that (i) d_n's are nondecreasing and (ii) $d_n \to \infty$.

Remarks. Notice that this does not imply that $\sigma_n(\varepsilon) \to \sigma(\varepsilon)$ for any particular ε. Neither can we conclude that σ or any of σ_n have limit at $\varepsilon = 0$.

5.7.1. *Summability.* This is the problem of finding for a sequence of functions σ_n a (canonical) function σ such that the sequence σ_n is its asymptotic expansion. As σ_n usually appears as a sequence of partial sums of a formal series $\sum_0^\infty f_n$, this question is called *summability*.

The *Watson-Nevalinna* theorem on Borel summability gives an affirmative answer for series $\sum c_n\varepsilon^n$, such that $\sum \frac{c_n}{n!}\varepsilon^n$ converges and the sum extends analytically.

5.7.2. *Feynman expansions and asymptotic expansions.* It would be nice if the perturbative expansions of a QFT would be an asymptotic expansion. This is known for the ϕ^4 theory using the Watson-Nevalinna theorem. For this reason we sometimes speak loosely of perturbative expansions of a QFT as "asymptotic expansions".

5.8. **Appendix B. Some points of view on renormalization.** We will use renormalization as a formal procedure that removes infinities.

However, physicists also have a more conceptual framework for renormalization as a flow in the space of possible values of all possible *couplings*, i.e., the dimensionless constants (parameters) of the theory. Here, renormalization has the effect (like a mass of a certain particle of shifting the *bare* values of couplings to the *effective* values. The bare values are those that one knows from other experiments, the *effective* values are the ones that "effectively" appear in a given situation, in other words if we plug in the effective values in formulas, we get the numbers actually observed in experiments. This shift of values is not mystical, it is simply the effect of some interactions which we prefer to keep out of the picture because we do not understand them well enough.

5.8.1. *Example: Hydrodynamics.* The formalism of renormalization can be trace to hydrodynamics to explain why acceleration of the ping pong ball in water as it rushes towards the surface. A naive computation predicts acceleration around 11 times the the gravitational acceleration g. When one analyzes the situation precisely it turns out that one formal way to deduce the correct answer is to replace in the naive computation the actual mass m of the ball (the "bare mass") with its "effective" version which is $m + \frac{1}{2}M$ where M is the mass of water that the ball can hold. This renormalization of the mass is just a formal way to account for (or sweep under the rug) the effect of the interaction of the ball and the water. (The actual acceleration is $< 2g$.)

5.8.2. *Mathematics.* The structure of a certain algorithm (HPBZ) for a specific incarnation of renormalization, the *minimal subtraction with dimensional regularization*, has been explained by Kreimer and Connes-Kreimer in terms of a Hopf algebra structure on graphs. This is further reformulated in terms of the Birkhoff factorization of loops (into positive and negative), the Riemann-Hilbert correspondence and a relation to motivic Galois groups [Connes-Marcolli].

5.8.3. *Several story lines.* All together, renormalization appears with seemingly disparate justifications:

- (i) Regularization of infinities (by any means, however disrespectful).
- (ii) A mechanism that explains variation of parameters in Lagrangian with the observation scale (for instance the variation of the charge we perceive when we vary the distance from the source of this charge, see 5.10.7).
- (iii) The "*physics depends on the scale*" principle. It appears in various guises, for instance, the fact that a given experiment has restricted energies available, so we are only seeing a part of the world that lives at these energies, not the true picture of the world.

5.9. **Appendix C. Renormalization in Statistical Physics.**

5.9.1. *Statistical Physics.* More recently, renormalization idea has been crucial in Statistical Physics. Here, the primary concern of renormalization is the *critical phenomena*.

The mechanism is the "*coarse graining*" or "*defocusing*" of short distance degrees of freedom, when long distance phenomena are of interest. This only affects the parameters of the system while keeping the interesting physical aspects the same. This transformation is called the Renormalization Group Flow (RGF) in the space of parameters (this is not a continuous flow, just an action of $(\mathbb{N}, +)$). Now one studies the RGF and what it says about the system.

5.9.2. *Coarse graining: the block spin transform.* This is a change of scale in the following sense. One considers the effect of interactions of charges placed on a grid. When one groups the points of the grid into, say, $a \times a$ blocks and moves further away, one may effectively perceive a new grid whose points are the blocks of the original one. The charge of a new point is the total effect of charges in the block.

However, this total effect often has simple approximations. Say, if charges (spins) are $s = \pm 1$ we assign to each new block a charge $s' = \pm 1$ by some rule that we call the *spin block transform*. For instance the majority rule (when a is odd).

Now we *rescale* the picture by a factor of a, i.e., consider the new system of charges s' on the new grid.

5.9.3. *RG flow on Hamiltonians.* We are interested in correlations of the system which are statistical averages over a large set of typical configurations. These are calculated in very much the same way as in QFT. When one passes from Feynman integrals to operator formalism the action S is replaced by the Hamiltonian operator H and the correlator formula goes from $\int e^{S(x)}\phi$ to $Tr[e^{-\beta H}\phi]$. The latter is also the form of the correlator in Statistical Physics, here the vector space has a basis of all possible configurations s of the system.

While the original system had hamiltonian H that applies to configurations s, the blocked system has a new Hamiltonian $\mathcal{H}'$ that applies to block configuration s'. The passage from H to H' is very much the same as the passage from $S[\Lambda]$ to $S[\Lambda']$ in QFT, the "integrating out" of degrees of freedom between two scales Λ, Λ' appears here as integrating out the *short* degrees of freedom (charges of points in the original grid) by taking the trace over all possible configurations in each block. (The reason is the same: the correlators must be invariant of the scale.)

5.9.4. *The Renormalization Group assumption.* One assumes that after any number of block transforms, the dominant interactions will always be short range (something like "nearest neighbor only"). This assumption (or really its consequences), has been verified experimentally, by numerical simulations and theoretically in solvable systems.

The assumption implies that the new Hamiltonian $\mathcal{H}'$ is supposed to have the same form as $\mathcal{H}$ (the same formula). The only thing that changes is the vector σ of couplings (parameters). This change $\sigma \mapsto \sigma'$ is the RG flow on the space of couplings.

One can iterate this to get a sequence of systems $\Sigma_0 = \Sigma, \Sigma_1 = \Sigma', \Sigma_2, ...$ with parameters $\sigma_0, \sigma_1 = \sigma_0', \sigma_2 = \sigma_1', ...$ Here, the emphasizes is on the *critical values* σ of parameters that correspond to *self-similar* systems, i.e., the fixed points σ of the renormalization flow on the space of parameters: $\sigma' = \sigma$. (For instance at a critical temperature the rescaling does not change the statistics of the system.) These critical values govern the "long term behavior" of the flow $\sigma_0, \sigma_1, ...$

5.9.5. *QFT and Statistical Physics.* The QFT analogue of critical values in Statistical Physics are the *renormalizable theories*, the theories that are approximately scale-invariant (up to logarithmic terms), for instance the conformal field theories. Again, these correspond to fixed points of the renormalization flow in the space of parameters (couplings). A strong version of this analogy: classical critical systems in Statistical Physics are in a certain limit equivalent to renormalizable euclidean quantum field theories (Wilson).

This has provided an explanation of the *Universality Phenomenon*: . "Many systems with different constituents and microscopic interactions exhibit the same critical behavior in the scaling limit." The point is that it was known on the QFT side that there is often only a handful (finite or countable) of theories with given symmetries.

This observation is said to have led to a unification/cross-fertilization of methods of (i) particle physics, (ii) statistical mechanics and (ii) condensed matter theory. One example is the application of conformal field theory (CFT) (which was first developed as a tool in string theory), to statistical mechanics and condensed matter physics.

The correspondence has strong numerical power in Statistical Physics – the renormalizable quantum field theories in two dimensions have been classified and in many cases they providing exact expressions for critical exponents, correlation functions, and other universal quantities in statistical systems.

5.10. **Appendix D. Renormalization of charge as interaction with vacuum.** This involves a beautiful picture that seems to be standard in physics. Besides explaining renormalization of charge of an electron in QED, this is also the mechanism of the Hawking radiation out of the black hole. The picture is of the effect of

Pairs of particles that appear from the vacuum and then disappear (cancel each other).

5.10.1. *Virtual particles.* We will need the point of view that there are two kinds of particles:

(1) *The usual ones.* These can exist forever. The reason is that they have their own substance like energy or momentum.

(2) *Virtual particles.* These are short lived because they have to borrow their substance from the vacuum and then they have to return it shortly.

Therefore, vacuum is really a rich structure. It is OK to think that it is a soup of particles that are balanced hence not perceived, except that when this "Dirac sea" of particles is disturbed by something like interaction of two particles, the soup bubbles and emits pair of particles which are now perceivable since they have a measurable effect on the usual particles. About vacuum one does not understand much more than this, so we summarize:

(1) This picture is satisfactory in the sense of theoretic computation. One can compute with it and (some) results agree with experiments (for instance for the charge renormalization).
(2) Philosophically, the picture is incomplete: who is vacuum that borrows and takes back?
(3) One measurable quantity related to the vacuum is the *vacuum energy* which is often called "the" *cosmological constant* (see 5.10.6). However, this leads to a *cosmological constant problem* – the measured values and theoretical predictions differ sharply.

5.10.2. *Vacuum mediation of particle interactions by virtual (anti)particles.* The idea is that when two particles meet, they interact through virtual particles emitted by the vacuum. We will consider it here in the path integral picture, i.e., in terms of Lagrangians and Feynman diagrams.

A standard example is when two ϕ-particles called 1 and 2, collide and emit two more ϕ-particles 3 and 4.

Let us draw this as a graph with one vertex (the collision) and 4 tails (the trajectories of two incoming and two outgoing particles). The time direction will be from time t_1 when we have particles $1, 2$ to time t_2 when we have $3, 4$.

However, the mediation picture says that the vertex v at which the interaction takes place, does not really exist. It is "smoothed out" by a small circle which seems to be the trajectory of a pair of two virtual "ψ" particles.

The four tails $1, 2, 3, 4$ meet the circle at four trivalent vertices $\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}$ divide the circle into four segments $\boxed{12}$, $\boxed{13}$, $\boxed{24}$, $\boxed{34}$. Now the interactions actually happen at the four vertices on this circle, we label the vertex at which the i-tail is attached by i. At each vertex one ϕ particle (a tail) meets two ψ particles, (two segments of a circle). This scenario of interaction is seen in the Lagrangian (action) as the $\phi\psi\psi$ term

$$\int \phi\psi\psi.$$

5.10.3. *The interaction.* The story say that the approaching ϕ-particles $1, 2$ disturb the vacuum and produce a pair of virtual particles ψ_1, ψ_2 which emerge at the center of the segment $\boxed{12}$. (One of ψ_1, ψ_2 is perceived as a particle and the other one as an antiparticle.)

Then ψ particles ψ_i $(i = 1, 2)$ travels to i on the segment $\boxed{12}$, in order to interact with the ϕ-particle ϕ_i. So, segment 12 is the picture of the emergence of the ψ particles.

The ϕ particles disappear when they meet the ψ particles(?). After ψ_1 reaches vertex $\boxed{1}$ it continues on the segment $\boxed{13}$ to the vertex $\boxed{3}$ and ψ_2 continuous on $\boxed{24}$ to the vertex $\boxed{4}$.

Here, ϕ-particles get reconstituted as $\boxed{3}$ and $\boxed{4}$, while ψ_i continue on the segment $\boxed{34}$ where they meet and annihilate each other.

5.10.4. *Temporary violation of conservation of momenta (energy).* This is one aspect of nonstandard physics in this story.

Let p_i be the momentum of particle i. The conservation of momenta (energy) in the time interval $[t_1, t_2]$ appears as a factor $\delta(p_1 + p_2 - p_3 - p_4)$, i.e., the ingoing total momentum $p_1 + p_2$equals the outgoing total momentum $p_3 + p_4$.

However, if we cut the timeline at some t' between t_1, t_2, then the conservation of momenta (energy) is violated on intervals $[t_1, t']$ and $[t', t_2]$. For the time interval $[t_1, t']$ the point is that whatever p_1, p_2 were, the momenta of ψ_i can be essentially arbitrary. The reason is that ψ_i "borrow" momenta (energy) from the vacuum. Then they have to return these on the time interval $[t', t_2]$.

5.10.5. *Singularity of the amplitude of this diagram.* The point that the momentum of ψ_i is arbitrary means that we have to integrate over all possible momenta $0 \leq p_i \leq \infty$. The four segments $\boxed{12}$, $\boxed{13}$, $\boxed{24}$, $\boxed{34}$ are four edges in the diagram (graph), each contributes a propagator P, so the integrand has the fourth power P^4 of the propagator. This makes the integral divergent.

So, one is forced to regularize the integrals with cutoffs, say $0 < \varepsilon \leq p_i \leq L < \infty$.

5.10.6. *The cosmological constant problem.* The energy of the empty space is the eigenvalue of the Hamiltonian H on the vacuum vector

$$H\ |0\rangle\ =\ E_{vac}\cdot|0\rangle\ .$$

The formula for the Hamiltonian is something like

$$H\ =\ \sum_{n\in\mathbb{Z}}\ p_n(a_n a_n^+ + a_n^+ a_n),$$

where a_n are the annihilation operators so they kill vacuum.

So, E_{vac} comes from the commutators $[a_n, a_n^+] = \frac{1}{2}$. This topically gives a constant (say, the light speed) times a sum of powers of natural numbers

$$E_{vac} \ = \ \sum_n \ n^k = \ c\zeta(-k).$$

The creation and annihilation of pairs is here reflected in the Hamiltonian formula

$$H \ = \ \sum_{n\in\mathbb{Z}} \ p_n(a_n a_n^+ + a_n^+ a_n).$$

A (Nobel) *Problem.* The vacuum energy predicted by computation is much larger than what is actually measured:

$$E_{vac}^{theory} \ \sim \ 10^{100} \ E_{vac}^{measurement}.$$

5.10.7. *Screening of the charge.* The renormalization of the charge is an effect of this interaction with the vacuum.

5.10.8. *Does the vacuum mediation smooth out the interaction?* We describe a collision of point sources in terms of a diagram (graph) which is still a singular object?

Remark. The choices of a complement to a subspace $U\subseteq V$ form a torsor for $\mathrm{Hom}(V/U, U)$ since any choice of a complement Σ gives a parametrization of complements by

$$\mathrm{Hom}(V/U, U) \cong \mathrm{Hom}(\Sigma, U) \ni A\mapsto\Gamma_A \stackrel{\mathrm{def}}{=} (id + A)\Sigma.$$

So, renormalization schemes form a torsor for $\mathrm{Hom}(\mathcal{P}/\mathcal{P}_{\leq 0}, \mathcal{P}_{\leq 0})$.

5.11. **Appendix. Graph weights and motives.** The reason why graph weights $w_{\varepsilon L}^{\gamma}(M, Q, I)$ are asymptotically periods is that they are periods for the standard choice of the data (M, Q, I) ($M = \mathbb{R}^n$ ...) and the $\varepsilon \to 0$ asymptotics is independent of M.

The corresponding algebraic variety U_γ is the complement in $\mathbb{C}[E_\gamma]$ (or $\mathbb{P}(\mathbb{C}[E_\gamma])$ of a hypersurface given by the Kirchoff graph polynomial K_γ, The corresponding motives m_γ are called Feynman motives.

Feynman motives are Tate motives for small graphs but not in general. The weights of Tate motives are multiple zeta values and the converse is implied by Grothendieck's conjecture.

The appropriate "Euler characteristic" $\chi_\gamma = \chi_{new}(U_\Gamma)$ behaves as a Feynman weight, i.e., $\chi_{\sqcup\gamma_i} = \prod \chi_{\gamma_i}$ (Aluffi-Marcolli). Then χ_γ is called a *motivic Feynman rule.*[41]

On the level of Feynman motives and motivic Feynman rules the divergence of Feynman integrals is approached through resolutions, deformation and interpretation via local Igusa functions.

[41] "Feynman rule" means (a formula for) a graph weight.

Y. $\boxed{ToFix}$

6. Z. Periods

Periods are integrals of algebraic differential forms over homology cycles. (Any such integral can be reduced to an integral of a top differential form.)

6.0.1. *Relation to motives.* If we choose representatives ω_i for a basis of de Rham cohomology and representatives α_j for a basis of singular homology. Then the period matrix Π_X

$$\int_{\alpha_j} \omega_i$$

is an isomorphism of de Rham and singular cohomology.

6.0.2. *Period matrix.* For X defined over $\mathbb{Q}$ the period matrix is defined as an element of

$$GL_n(\mathbb{Q})\backslash M_n(\mathbb{C})/GL_n(\mathbb{Q}).$$

Example. For $X = \mathbb{P}^n$ the periods are $1, 2\pi i, ..., (2\pi i)^n$. (The period matrix is diagonal and the these are the diagonal entries?)

6.0.3. *Grothendieck's conjecture.* For X defined over $\mathbb{Q}$, the period matrix determines the $\mathbb{Q}$-motive M_X of X.

Example. The motive M_X of $X = \mathbb{P}^n$ splits into $n+1$ motives and their periods are $1, 2\pi i, ..., (2\pi i)^n$. (The period matrix is diagonal and the these are the diagonal entries?)

6.0.4. *Tate motives and multiple zeta values.* The periods of Tate motives are *multiple zeta values*. Grothendieck's conjecture implies

Corollary. The only $\mathbb{Z}$-motives whose periods are multiple zeta values are Tate motives.

Question. Should the t-structure on motives be in in terms of periods or period functions? (The question does not make any sense. Anyway.)

6.1. **Etale cohomology of $\mathbb{Q}$-varieties.** It carries an action of $G_{\mathbb{Q}}$. With this structure we can think of it as a realization of a motive. Then the analogues of periods (?) are the integers N_p, the count of the number of points of the reduction mod p.

6.1.1. *Tate conjecture.* The numbers $N_p(X)$ for almost all p determine the motive of X.

Example. For Tate motives N_p's are polynomials in p. The Tate conjecture implies the converse:

Corollary. The only $\mathbb{Z}$-motives such that N_p is polynomial in almost all p are mixed Tate motives.

Remark. This is a nice way to check whether the motive M_X of a given variety is a Tate motive, i.e., whether its periods are multiple zeta values: calculate $N_p(X)$'s, see whether they are polynomials of p.

6.2. **Multiple zeta values.** The *multiple zeta function* is defined as

$$\zeta(s_1, ..., s_k) \;=\; \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{n_1^{s_1}\cdot\cdot\cdot n_k^{s_k}}$$

The *multiple zeta values* are values at integers $(s_1, ..., s_k) \in \mathbb{Z}^k$, with $z_1 > 1$.

Lemma. Multiple zeta values are periods. Explicitly,

$$\zeta(s_1, ..., s_k) \;=\; \int_{1>t_1>\cdots>t_k>0} \omega_{s_1}(t_1)\cdot\cdot\cdot\omega_{s_k}(t_k)$$

where,

$$\omega_0 = \frac{dt}{t},\ \omega_1 = \frac{dt}{1-t} \quad \text{and} \quad \omega_r = \omega_0^{r-1}\wedge\omega_1 \ \text{ for } \ r \geq 2.$$

6.2.1. *The hypersurfaces X_Γ in $\mathbb{A}^{E_\gamma}$.* To a graph Γ one associates the Kirchoff polynomial in indeterminates x_e indexed by edges $e \in E_\gamma$

$$K_\Gamma \;\stackrel{\text{def}}{=}\; \sum_{T\in \mathrm{SpTr}_\Gamma} \prod_{e\in E_\Gamma - E_T} x_e.$$

The summands are monomials corresponding to the *spanning trees* of Γ.

No let X_Γ be the hypersurface in $\mathbb{A}^{E_\gamma}$ given by K_γ.

Kontsevich conjectured that the motive of X_Γ is Tate, i.e., that the periods are multiple zeta values. This was disproved by Belkale-Brosnan (though it is true for small graphs).

6.3. **Algebro geometric Feynman rules (Aluffi-Marcolli).** A *Feynman rule* is a formula for the weights $\gamma\mapsto w^\gamma$ of graphs.

The point of view here is that the correct variety associated to Γ is

$$U_\Gamma \;\stackrel{\text{def}}{=}\; \mathbb{P}(\mathbb{C}[E_\gamma]) - X_\Gamma.$$

The result is that certain invariant χ_{new} has the usual multiplicativity property of Feynman weights

$$\chi_{new}(U_\Gamma) \;=\; \prod \chi_{new}(U_{\Gamma_i}).$$

This is nontrivial since on the level of varieties the Euler characteristic $U_{\Gamma_1\sqcup\Gamma_2}$ is a G_m-bundle over $U_{\Gamma_1}\times U_{\Gamma_2}$. (In particular the Euler characteristic χ does not work.)

6.3.1. *Connes-Kreimer.* Feynman rule is a character of the Hopf algebra $\mathcal{H}$ of Feynman graphs. Namely, the collection of dimensionally regularized Feynman integrals $U(\Gamma, p)$ of all the 1PI graphs of a given scalar quantum field theory defines a homomorphism of unital commutative algebras

$$\phi \in Hom(\mathcal{H}, \mathcal{K})$$

where $\mathcal{K}$ is the field of germs of meromorphic functions at $z = 0 \in \mathbb{C}$.

The coproduct in the Hopf algebra is then used in to obtain a recursive formula for the Birkhoff factorization of loops in the pro-unipotent complex Lie group

$$G(\mathbb{C}) = Hom(H, \mathbb{C}).$$

This provides the counterterms and the renormalized values of all the Feynman integrals in the form of what is known in physics as the *Bogolyubov recursion*, or *BPHZ renormalization procedure*.

In particular,

(1) any character of the Hopf algebra $\mathcal{H}$ can be thought of as a possible assignment of Feynman rules for the given field theory, and
(2) the renormalization procedure can be applied to any such character as to the case of the Feynman integrals.

The characters need not necessarily take values in the field $\mathcal{K}$ of convergent Laurent series for the BPHZ renormalization procedure to make sense.

In fact, it was shown in how the same Connes-Kreimer recursive formula for the Birkhoff factorization of loops continues to work unchanged whenever the target of the Hopf algebra character is a *Rota-Baxter* algebra of weight $\lambda = -1$.[42]

In the Connes-Kreimer case, a Rota-Baxter operator it is the operator of projection of a Laurent series onto its divergent part (a renormalization scheme!). The Rota-Baxter identity is what is needed to show that, in the Birkhoff factorization = (œwith S the antipode and the product dual to the coproduct, the two terms $\phi_\pm$ are also algebra homomorphisms.

When working in the algebro-geometric world of the graph hypersurfaces X_Γ, one would like to have motivic Feynman rules, namely an assignment of an "Euler characteristic"

$$\chi_{new}$$

(the terminology views the class in the Grothendieck ring of varieties as the universal Euler characteristic); to the graph hypersurface complements

$$\mathbb{P}^{n1} - X_\Gamma.$$

[42]A Rota-Baxter ring of weight λ is a commutative ring R endowed with a linear operator $T : RR$ satisfying the Rota-Baxter identity

$$T(x)T(y) - \lambda T(xy) \;=\; T(xT(y)) + T(T(x)y).$$

Such an operator is called a Rota-Baxter operator of weight λ.

This should have the usual multiplicativity property of Feynman weights: $\Gamma = \sqcup\, \Gamma_i$ implies that

$$\chi_{new}(\mathbb{P}^{n-1} - X_\Gamma) \;=\; \prod\; \chi_{new}(\mathbb{P}^{n_i-1} - X_{\Gamma_i}).$$

Here the graph hypersurface X_Γ is defined as the hypersurface in $\mathbb{P}(\mathbb{C}[E_\gamma])$ given by the vanishing of the Kirchoff polynomial $K_\Gamma = \boxed{\Psi_\Gamma}$ (in general the sum is over spanning forests T rather than spanning trees). K_Γ is multiplicative: $K_{\sqcup\ \Gamma_i} = \prod\ K_{\Gamma_i}$.

The usual Euler characteristic does not satisfy the desired property. In fact, if Γ is not a forest, then for $\Gamma = \Gamma_1 \sqcup \Gamma_2$, $\mathbb{P}^{n1} - X_\Gamma$ is a G_m-bundle over the product $\prod_{i=1,2}\ \mathbb{P}^{n_i 1} - X_{\Gamma_i}$, hence its Euler characteristic vanishes.

The main result of the present paper is to show that a modification χ_{new} which corrects this problem exists in the Grothendieck ring of immersed conical varieties. It descends to a class in the Grothendieck ring of varieties, and to a Chern-Schwartz-MacPherson characteristic class of singular algebraic varieties,

Example. Motivic Feynman rule

$$U(\Gamma) \stackrel{\text{def}}{=} [\mathbb{A}^{E_\Gamma} - X_\Gamma]\mathbb{L}^{-n} \;\in\; K^0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}].$$

1 In this ring we can still consider the Rota-Baxter operator of projection onto the polar part in the variable $\mathbb{L}$.

Now, the renormalized Feynman rule is given by the universal CK formula of Birkhoff factorization.

Example. Consider the basis of $\mathbb{Q}[T]$ as a $\mathbb{Q}$-vector space, given by the polynomials

$$\pi_0(T) = 1 \quad \text{and} \quad \pi_n(T) = \frac{T(T+1)(T+n1)}{n!} \quad \text{for} \quad n > 0.$$

The shift operator in this basis

$$T(\pi_n) \;=\; \pi_{n+1}$$

is a non-trivial Rota-Baxter operator of weight -1. One can then apply the BPHZ procedure with respect to this operator.

6.4. **Divergences and renormalization (Aluffi-Marcolli).** Their analysis shows that when convergent, the parametric Feynman integral for (certain class of) graphs is a period of a mixed Tate motive. The technique is to show that certain relative cohomology is a realization of a mixed Tate motive $m(X, Y)$, where the loci X and Y are the complement of the determinant hypersurface and the intersection with this complement of a normal crossing divisor that contains the image of the boundary of the domain of integration σ_n under the map τ_Γ , for any graph Γ with fixed number of loops and genus. Knowing that m(X, Y) is a mixed Tate motive implies that,

They sketch possible approaches to divergences in the Feynman weights.

6.4.1. *Blowups.* [Bloch-Esnault-Kreimer] One can proceed to perform a series of blowups of strata of a certain intersection until one has separated the domain of integration from the hypersurface and in this way regularized the integral.

6.4.2. *Dimensional regularization and L-functions.* [Belkale-Brosnan] showed that *dimensionally regularized Feynman integrals* can be written (when they converge), in the form of a *local Igusa L-function*, where the coefficients of the Laurent series expansion are periods, (Or log-divergent?)

6.4.3. *Deformations.* An alternative to the use of blowups is the use of deformations.

7. XXX

Part 3. Chapter 1. Intro

0.1. **Goals.** The book lays complete foundations of PQFT (perturbative QFT) based on

(1) Feynman integral approach (including having an action S).
(2) Low energy effective actions $S[\Lambda]$ approach of Wilson.[43]

Actually, in order to understand the meaning of locality at each scale, instead of "energy effective action" we use "wordline length effective action".

0.1.1. *Perturbative limitation/aspect.* The central limitation of Costello's formalism is that

(1) The spaces of fields $\mathcal{E}$ are vector spaces and he only considers infinitesimally small fields.
(2) The Planck constant $\hbar$ is an infinitesimally small (i.e., formal) parameter, and at $\hbar = 0$ we get the classical limit.

- Here (1) appears through the assumption that the observables, i.e., functions on fields, are formal power series:

$$\mathcal{O}(\mathcal{E}) \stackrel{\text{def}}{=} \widehat{S}^{\bullet}(\mathcal{E}^{\vee}).$$

  This class of functions is the technical framework of the book.
- The meaning of infinitesimally small fields is that these are perturbations of a given field.

  The fields of the "original" theory can actually be maps $M \to X$. The perturbative version of this theory deals with perturbations Φ of one chosen solution ϕ_0 of the EL-equation. So, the fields are infinitesimal

  Passing to Costello's setting actually involves a choice of a linearization of the theory at ϕ_0, which replaces X with the ϕ_0-pull-back of the tangent bundle of X. Then the perturbations Φ of a solution ϕ_0 are of the form $\Phi = \phi_0 + \phi$ for small sections ϕ of $T_{\phi_0} Map(M, X) = \phi_0^* TX$. These are the fields that lie in the vector space $|Ga(M, \phi_0^* TX$.

Remark. Later Costello finds a way to circumvent the restrictions for gauge theory (via the BV-formalism, i.e., the QME), and also in cases when the structure on the target X can be described as a certain symmetry of X – a cdg-Lie algebra $\mathfrak{g}_X$ (holomorphic Chern-Simons).

0.2. **The space of quantizations of a classical theory.**

[43] $S[\Lambda]$ is also denoted $S^{eff}[\Lambda]$.

0.2.1. *Data.*

- Space of fields $\mathcal{E}$.
- The classical action S (a local functional on $\mathcal{E}$)

Let $\mathcal{T}^{(n)}(\mathcal{E}, S)$ be the space of *quantizations modulo* $\hbar^{n+1}$ of the classical theory. We are interested in $\mathcal{T}^{(\infty)}(\mathcal{E}, S) = \lim_{\leftarrow} \mathcal{T}^{(n)}(\mathcal{E}, S)$.

Theorem. $\mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)}$ is a torsor for the abelian group of local action functionals.

Question. $\mathcal{T}^{(\infty)}(\mathcal{E}, S)$ should then itself be a torsor for some huge self-extension of local action functionals $\mathcal{O}_l^{\infty}(\mathcal{E}, S) = \lim_{\leftarrow} \mathcal{O}_l^{(n)}(\mathcal{E}, S)$.

0.2.2. *β-functions.* They appear as coefficients of the RGF action on a given Lagrangian. (Computed in examples.)

0.2.3. *Renormalizable theories.* This involves the *renormalization group flow* (RGF) on the space of theories. This is essentially the rescaling of the spacetime $\mathbb{R}^n$.

0.2.4. *Perturbatively renormalizable theories.* A theory is perturbatively *renormalizable* if it has a **critical** scaling behavior, i.e., the RGF flow fixes it modulo the logarithmic corrections.

Theorem. These are classified.

The method of construction of renormalizable theories (that satisfy the QME if we are in the BV formalism) is again cohomological, i.e., obstruction theoretic as above. It provides existence when obstruction cohomology group vanishes and then the space of quantizations is a torsor for another cohomology group.

Remark. Here renormalizability means the vanishing or a calculation of a certain cohomology group. Traditionally it involves Feynman diagram manipulations.

0.3. **Example: Gauge theory.** GT is made to fit this formalism by combining effective action idea and the BV formalism.

Here gauge symmetry of the family $S[\Lambda]$ is expressed as: $S[\Lambda]$ satisfies the **scale Λ QME** .

0.3.1. *Renormalizability of pure YM.* It is proved by a calculation of Gelfand-Fuchs cohomology. ABBA

1. Feynman integrals

Feynman's formulation of QFT in terms of functional integrals says that the actual universe is a quantum superposition ("statistical superposition ") of all possible configurations, i.e., fields.

1.1. Lorentzian (physical) version.

Here M is a manifold with a Lorenz signature.

Each field is one description of the physical system ("the universe"). Our basic example is the *single scalar field theory*, i.e., the fields are $\mathcal{E} = C^\infty(M)$.

1.1.1. *The action.* The action S is an integral over M of the Lagrangian L:

$$S(\phi) \ = \ \int_{x\in M} \ \mathsf{L}(\phi)(x).$$

The Lagrangian L is a sum of the free part L_{free} and the interaction part I.

Example. For a massive field the free part is

$$\mathsf{L}_{free} = \ \phi(\Delta + m^2)\phi$$

A typical interaction is

$$\mathsf{I}(\phi) = \ \phi^4.$$

1.1.2. *Observables.* These are the measurements one can make. The result depends on the state of the system, so observables are functions on fields

$$\mathrm{Ob}(U) = \ \mathcal{O}(\mathcal{E}(U)).$$

The most standard class are the observations that can be made by an observer at the point $x \in M$. First, at $x \in M$ we have observables $\mathcal{O}_x \stackrel{\mathrm{def}}{=} ev_x$, the evaluation of the field ϕ at x, More generally, we also have the derivatives of ϕ at x, i.e., the observables $\mathcal{O}_{x,D} = ev_x(D\phi)$ for differential operators D.

1.1.3. *Correlators of a family of observables.* These are integrals over fields

$$\langle O_1, ..., O_n\rangle \stackrel{\mathrm{def}}{=} \int_{\phi\in\mathcal{E}(M)} \mathcal{D}\phi \ e^{iS(\phi)/\hbar} \ O_1(\phi)\cdot\cdot O_n(\phi), \qquad O_i \in \mathrm{Ob}.$$

They measure the relation between quantities in the theory, i.e., various measurements.

1.1.4. *The Problem.* There is no natural measure $\mathcal{D}\phi$ on the space of fields, we only imagine it. When $\mathcal{E}(M)$ is a vector space, we would like it to be a Haar measure.

1.2. **Euclidean version of QFT: a statistical theory.** Here M is an Euclidean manifold.[44] This setting gives

(1) A statistical interpretation of Feynman integrals.
(2) A large supply of manifolds.
    This allows calculations by cutting and pasting. These are not physical as they do not have Lorentzian analogues.

Fields and Lagrangians are as before except that instead of a Minkowski Laplacian we now have an Euclidean Laplacian. The correlator integral is changed by replacing $iS/\hbar$ by $-S/T$.

1.2.1. *Probability and Temperature.* Here $T = -i\hbar$ is interpreted as **temperature of the system** and $e^{-S(\phi)/T}$ as probability (up to an overall normalization factor Z) that the system be in the state ϕ.

1.2.2. *Perturbative aspect.* In the Euclidean picture it means that

- as $T \to 0$, the universe freezes into one classical solution ϕ_0;
- we only consider what happens for infinitesimally small T (a formal variable), so fields ϕ only vary infinitesimally away from ϕ_0.

The same happens in the Minkowski picture. For $\hbar = 0$ the world is classical, i.e., described by a solution ϕ_0 and we consider $\hbar$ as a formal variable.

2. Wilsonian strategy of low energy theories

The "low energy" concept will be first described in terms of the spectral analysis of the Laplacian. This is intuitive but not local. Then it will be (re)formulated in terms of the length of worldlines because this approach is better suited for describing the meaning that locality acquires when one considers different scales.

2.0.3. *Transition to the worldline view.* The switch from energy to length involves passing from formal (i.e., non defined) functional integrals over fields to their well defined perturbative expansions (by applying Wick lemma formally). These perturbative expansions are sums indexed by Feynman graphs but have a geometric interpretation as integrals over the moduli of evolutions of finite systems of particles. So, the switch involves passing from energy of fields in QFT to worldline length of particles (a kind of QM).

2.1. **Effective actions $S[\Lambda]$ at the energy scale $\leq \Lambda$.** This part is the historical introduction to the idea of effective QFT. Later we will work in the length scale.

[44] The book works in Euclidean formalism.

2.1.1. *Decomposition of fields according to the energy.* Here the harmonic analysis of the Laplacian operator on the space of fields $\mathcal{E}$ represents it as an integral

$$\int_0^{+\infty} \mathcal{E}(\Lambda)\ \partial\Lambda$$

of eigenspaces of the Laplacian. So, we have summands

$$\mathcal{E}_{\leq\Lambda}\ \subseteq\ \mathcal{E}.$$

2.1.2. *Observables $Ob_{\leq\Lambda}$ of energy $\leq\Lambda$.* This embeds the "energy $\leq\Lambda$-observables"

$$\mathrm{Ob}_{\leq\Lambda}\ \stackrel{\mathrm{def}}{=}\ \mathcal{O}(\mathcal{E}_{\leq\Lambda})$$

into all observables Ob, via the projection $\mathcal{E}\twoheadrightarrow\mathcal{E}_{\leq\Lambda}$.

The strategy of considering only the phenomena of energy $\leq\Lambda$ is

- (i) realistic as this is what we can observe in any given experiment,
- (ii) it kills the infinities in QFT.

So, infinities only arise when we think of the classical Lagrangian S as adequate for all energy scales including the infinite one. Wilson's insight was that *physics depends on the scale*, i.e., that what governs the behavior at scales i$\leq\Lambda$ is not the classical Lagrangian but the "scale Λ Lagrangian $S[\Lambda]$ (the "effective" Lagrangian), which one has to deduce from S

2.2. **Renormalization group flow: the length scale.** Recall the meaning of the transition to the worldline view from 2.0.3.

Passing from the energy scale Λ to the worldline length L is roughly by

$$L\sim\frac{1}{\Lambda}.$$

("Energy of a vibrating string is higher if the length of waves is lower.")

Therefore, "energy $\leq\Lambda$" corresponds to "length $\geq L$". So, the effective action $S[L]$ will now describe phenomena that involve lengths $\geq L$. The high energy limit $\Lambda\to\infty$ corresponds to the low length limit $L\to 0$.

In the setting of the length scale the action at length scale $\geq L$ is denoted $S[L]$ and RG flow is denoted $W_{\varepsilon,\Lambda}$ where usually $\varepsilon\leq L$. Now the main step in the quantization of classical theories is to make sense of $W_{0,L}=\lim_{\varepsilon\to 0} W_{\varepsilon,L}$.

2.2.1. *Propagators.* The main purpose of the perturbative expansion of the partition function (or of the RGF flow W) over (stable) Feynman graphs is to deal with the "non-free" part of the action $I=S-S_{free}$ which in the worldline picture describes interactions between particles. The weight associated to a particular graph is calculated combinatorially using the propagator.

The free part of the Lagrangian is $\phi Q\phi = \langle \phi, Q\phi \rangle$ for an invertible quadratic operator Q.[45]

While Q appears in the Feynman functional integral, the incarnation of Q that figures in its perturbative expansion is the propagator P. Propagator itself is an integral of the heat kernel K of Q which is the most fundamental object in the analysis which we will use.

Propagator $P(x, y)$ is a distribution on M^2 which is a smooth function off the diagonal. Here are several views on $P(x, y)$

(1) It is the integral kernel for Q^{-1}.
(2) $P(x, y)$ measures the correlation of values of fields at points $x, y \in M$ in the free theory given by Q.
(3) P is an integral of the heat kernel K of Q

$$P \;=\; \int_{\tau=0}^{\infty} d\tau \;\; K_\tau.$$

(4) $P(x, y)$ is a functional integral over the spaces $\mathcal{P}(x, y)$ of paths, i.e., worldlines of a particles that travel from x to y. (Here, $\mathcal{P}_l(x, y) \subseteq \mathcal{P}(x, y)$ are the paths that take time l.)

$$P(x, y) \;=\; \int_{l=0}^{\infty} dl \;\; \int_{f \in \mathcal{P}_l(x,y)} \mathcal{D}f \;\, e^{-E(f)};$$

the action is the energy of the path $E(f) = \int_0^l \; |df|^2/2$.

Remark. The heat kernel $K_l(x, y)$ is the integral kernel for e^{-lQ}. So, formula (4) follows from (3) and the functional integral interpretation of the heat kernel:

$$K_l(x, y) \;\stackrel{\text{def}}{=}\; \int_{f \in \mathcal{P}_l(x,y)} \mathcal{D}f \;\, e^{-E(f)}.$$

2.2.2. *Worldline length or "proper time".* The parameter l in Feynman's formula (3) will be called the worldline length. Its physical interpretation is as *proper time* – the time on the worldline, i.e., the time measured by the clock traveling on the worldline. (It is not related to the time on the spacetime.)

2.2.3. *The perturbative expansion over Feynman graphs of the interaction part of the action.* From the wordline point of view ("particles in the spacetime"), the quantity $I(\phi)$ describes how particles interact.

These interactions are described by Feynman graphs because these graphs are the "**world-graphs**" of a family of particles traveling in spacetime (at random). The information in

[45] For the massive scalar field theory $Q = \Delta + m^2$.

the world-graph is obvious – these are drawings of all ways for particles to come together and to break apart.

2.2.4. *Relation to strings.* In String Theory. particles are not points but (small) loops, so their worldgraphs are not Feynman diagrams but their thickenings – the 2d surfaces.[46]

2.3. **Renormalization group flow as a sum over Feynman graphs.** The RG flow consists of operators $W_{\varepsilon,L}$ on the space $\mathcal{I}_q \subseteq \mathcal{O}_l(\mathcal{E})[[\hbar]]$ of quantum interactions.

Here we define flow operators using just linear algebra. The basic elements of this formalism are the *weights* $w^\gamma_{\varepsilon L}$ of Feynman graphs γ. The weight $w^\gamma_{\varepsilon L}$ is an operator

$$w^\gamma_{\varepsilon L} : \mathcal{I}_q \to \mathcal{O}^{d_\gamma}(\mathcal{E}) \subseteq \mathcal{O}(\mathcal{E})$$

where d_γ is the external degree of the graph γ. It is defined as a contraction of two tensors

$$w^\gamma_{\varepsilon L} I \stackrel{\text{def}}{=} \langle P_\gamma, I_\gamma \rangle.$$

Now, $W_{\varepsilon L}$ is an integral over the moduli space of $cs\mathcal{FG}$ of connected stable Feynman graphs

$$W_{\varepsilon L} \stackrel{\text{def}}{=} \int_{\gamma \in cs\mathcal{FG}} \hbar^{g_\gamma} w^\gamma_{\varepsilon L}$$

where g_γ is the genus of the Feynman graph γ.

xx

First, they have expansions

$$W_{\varepsilon L} \stackrel{\text{def}}{=} \sum_{i,k \geq 0} \hbar^i \, W(\varepsilon, L)_{ik}$$

in the sense that for $I \in \mathcal{I}_q$

$$W_{\varepsilon L ik} I \stackrel{\text{def}}{=} (W_{\varepsilon L} I)_{ik} \quad \text{hence} \quad W_{\varepsilon L} I \; = \; \sum_{i,k} \hbar^i W_{\varepsilon L ik} I.$$

The ik-coefficient is an integral over the moduli space of $cs\mathcal{FG}_{ik}$ of connected stable Feynman graphs γ of genus g and external degree ("external valency") k,

$$W_{\varepsilon L ik} \; = \; \int_{\gamma \in cs\mathcal{FG}_{ik}} w^\gamma_{\varepsilon L}.$$

yy

(1) The (inner) edges of γ are labeled by the (ε, L)-propagators

$$P_{\varepsilon, L} \stackrel{\text{def}}{=} \int_{|tau=\varepsilon}^{L} d\tau \; e^{-m\tau^2} \; K_\tau.$$

[46] An impressive ingredient of this parallel is the use of stable Feynman graphs and stable curves.

(2) the vertices of valency k (and genus i) labeled by the coefficient $I_{i,k}[\varepsilon]$ in the expansion of

$$I[\varepsilon](\phi,\hbar) \;=\; \sum_{i\geq 0} \hbar^i I_k[\varepsilon](\phi) \;=\; \sum_{i\geq 0} \hbar^i \sum_{k\geq 0} I_{i,k}[\varepsilon].$$

Here $I[\varepsilon](\phi)$ is a function of $\hbar, \phi$, so the coefficient $\sum_k I_{ik}[\varepsilon](\phi)$ of $\hbar^i$ is a function of the field ϕ. In a perturbative theory fields live in a vector space (sections of a vector bundle) and then $I_{i,k}$ is just the homogeneous component of I_k of degree i in ϕ.

Here valency k of a vertex means that k-particles collide at this vertex (some of them incoming and the rest outgoing).

Remark. For the meaning of (3) see the picture 1 on page 19.

2.3.1. *Summary of length scale formulations of RGF.* We will use the phrase

"allow particles to travel between interactions for the length in interval J.

It means that the integral involved is over worldlines f "whose length scale" is in J, meaning that the length of each leg is in J.[47]

The case $J = (L, \infty)$ appears for correlator integrals of observables O_i that have "length scale $\geq L$", meaning that each O_i naturally defined on worldlines of length scale $\geq L$.

The case $J = [\varepsilon, L)$ appears for the renormalization flow integrals $W_{\varepsilon,L}$ where produce $S[\varepsilon]$ from $S[L]$ by integrating out the worldlines with length scale between ε and L.

Here are some formulations of RGE:
1. *The correlator formulation.* A family $I[L]$ of quantum interactions satisfies RGE if the correlators are independent of L. More precisely, the $I[L]$-correlator of observables on the length scale $\geq M$ is independent of L as long as $L \leq M$. A restatement:

"If we allow particles to travel for at least length M between interactions. then then for $L \leq M$ the $I[L]$-correlator does not depend on the choice of L."[48]

2. *The action formulation.* For $\varepsilon < L$, $I[L]$ is obtained from $I[\varepsilon]$ by integrating out the worldlines whose proper time is between ε and L. Restatement:

"$I[L]$ can be obtained from $I[\varepsilon]$ by allowing particles to travel between interactions along paths whose proper time is in $[\varepsilon, L)$, and then interact using $I[\varepsilon]$."

3. *The Feynman graph formulation.* One obtains $I[L]$ from $I[\varepsilon]$ by applying operator $W_{\varepsilon,L}$ which is a sum over Feynman graphs.

[47] A "leg" means here an internal edge of the metrized graph on which map f is defined.

[48] One also says this with one variable: "if we allow particles to travel for at least length L between interactions. then then the $I[L]$-correlator does not depend on the choice of L."

3. Wilson's definition of QFT in terms of effective actions: the energy scale formulation

Wilson defines a *continuum QFT* as a family of "effective actions" $S[\Lambda]$ indexed by energy scales $0 < \Lambda < \infty$ and satisfying the following principles:

(1) [*Physics depends on the scale.*] Each $S[\lambda]$ is a function on $\mathrm{Ob}_{\leq\Lambda}$ the observables of energy $\leq \Lambda$. It is a power series in $\hbar$ and in the field ϕ.
(2) [*Classical limit.*] The classical part (i.e. the image modulo $\hbar$) $S[\Lambda]|_{\hbar=0}$, is of the form
$$S[\Lambda](\phi) \;=\; \int_M I(\phi) \;-\; \phi Q \phi$$
where the free part is given by an invertible hermitian linear operator Q,[49] and the interaction part $I(\phi)$ is at least cubic.[50]
(3) [*Renormalization Group Equation.*] For $\Lambda' < \Lambda$, $S[\Lambda']$ is determined from $S[\Lambda]$ by the RGE.
(4) [*Locality.*] $S[\Lambda]$ satisfies a Λ-locality axiom formulated below in 4.2.3.

4. Locality Principle in EQFT: the length scale formulation

4.1. Principle of locality.

Roughly it says that

Interactions between fundamental particles only happen at points.

In other words, particles only interact when they are at the same point of space. This includes the principle

NO *spooky action at a distance.*

In other words if particles interact at a distance this is through the medium of other particles between them.

In terms of the action the locality principle appears as the requirement that the Lagrangian is a *local functional* on fields. This means that it is a function on the jet bundle, i.e., a finite sum of products $D_i\phi$ for differential operators D_i.

In terms of the effective action formalism, the locality is is not really an absolute principle – it holds only if we include all scales. So the at any given range of scales it only holds approximately and the approximation gets better as we allow larger range. A precise formulation is that locality holds asymptotically (the definition of asymptotic expansions is in 5.7 below).

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[49] For instance $Q = \Delta + m^2$ in the massive scalar field theory.
[50] We may write this as $I = O(\phi^3)$.

4.2. **Locality in EQFT: the notion of** *asymptotic locality***.** We will see that the formulation of the locality principle of EQFT in the language of energy scale is not satisfactory, so we will restate it in terms of the length scale.

4.2.1. *Energy scale.* Our first attempt is to say that a solution $S[\Lambda]$ of RGE is *asymptotically local* if it has an asymptotic expansion for *large* Λ, as a linear combination of local functionals Θ_i, with coefficients the function of the scale Λ :

$$S[\Lambda](\phi) \stackrel{AE}{=} \sum_{i=0}^{\infty} f_i(\Lambda)\ \Theta_i(\phi).$$

This formalism is not compatible with the RGF action – if $S[\Lambda]$ is close to local then RGE forces $S[\Lambda']$ to be "completely nonlocal".

4.2.2. *Length scale.* We say that a solution $S[L]$ of RGE is *asymptotically local* if it has an asymptotic expansion for *small* L, as a linear combination of local functionals Θ_i, with coefficients the function of the length scale L :

$$S[L](\phi) \ \cong\ \sum_{i} f_i(L)\ \Theta_i(\phi).$$

4.2.3. *A (provisional) complete definition of effective QFT (combination of energy and length scale viewpoints).* Now we complete the definition by combining the use of energy scale (for RGE) and the length scale (for locality).

A continuum QFT is a family of "effective actions" $S[\Lambda]$ indexed by energy scales $0 < \Lambda < \infty$ and satisfying the properties (1-3) above as well as the

- (4) [Locality] When $S[\Lambda]$ is translated into the length scale effective action $S[L]$ then $S[L]$ satisfies the *asymptotic locality* axiom.

5. The classification of effective Quantum Field Theories

Now that we have the notion of effective Quantum Field Theories (4.2.3), we can reformulate the above mentioned theorem that classifies quantizations of a particular classical theory, i.e., classical action S, The new version is a classification of all effective Quantum Field Theories.

Let us fix a given space of fields $\mathcal{E}$, i.e., a choice of (M, g, E), Denote by $\mathcal{T}^{(n)}$ the space of Effective Quantum Field Theories which are *defined modulo* $\hbar^{n+1}$.[51]

[51]This is just the version of the above notion of EQFT which uses the ring $\Bbbk[\hbar]/h^{n+1}$ instead of $\Bbbk[[\hbar]]$.

Theorem. The "forgetting the $\hbar^{n+1}$ term" map

$$\mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)},$$

is a torsor for the abelian group of local action functionals $\mathcal{O}_l(\mathcal{E})$.[52]

Remark. We can identify the space of theories (unnaturally) with the space of $\hbar$-series in local action functionals – but only if we choose a section for each torsor $\mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)}$. The standard way to do this is to choose a *renormalization scheme*.

5.0.4. *Renormalization schemes.* A renormalization scheme is a way to extract the singular part of certain functions of one variable:

(1) The first observation is that we are only interested in $\mathcal{P} \subseteq C^\infty(0,\infty)$ of *period functions*.[53] Let $\mathcal{P}_{\leq\infty} \subseteq \mathcal{P}$ be the subalgebra of functions which have limit at ∞.
(2) A renormalization scheme is is a choice of a complementary subspace Σ.

5.0.5. *The use of a renormalization scheme to parametrize effective QFTs.*

Theorem. A choice of a renormalization scheme Σ provides a bijection between the space $\mathcal{T}^{(\infty)}$ of all EQFTs and the space $\mathcal{I}_q$ of all local quantum interactions I (local functionals with values in $\Bbbk[[\hbar]]$ such that the constant term is at least cubic).

Remark. We can state this in terms of the local Lagrangians $\mathcal{L}$ which give local action functionals $S = \int_M \mathcal{L}$. The space of theories is in bijection with $\Bbbk[[\hbar]]$-valued local Lagrangians modulo Lagrangians which are total derivatives.

6. YYY

Part 4. Chapter 2. Effective Theories, Scaled actions and Counterterms

Here we define EQFTs and prove that the quantization procedure is a torsor (term by term) over local action functionals.

0.0.6. *Data for a classical theory.* The data for a classical theory are

- A Riemannian manifold (M, g).
- A vector bundle E whose sections form the space of fields $\mathcal{E}$.
- A functional $S \in \mathcal{O}(\mathcal{E})$ which is a *(classical) action functional*, i.e.,
  (1) 0 is a critical point with value zero and
  (2) S is local.

[52] Torsor terminology here means a "torsor that has a section", i.e., a trivial or "non-empty" torsor. So, the claim is that there are no obstructions.

[53] $\mathcal{P}$ is related to variations of Hodge structures.

Action functionals decompose into the quadratic term and higher terms $S = S_2 + I$ where

(1) In terms of the inner product $\langle -, - \rangle = \langle -, - \rangle_M$ on $\mathcal{E}(M) = C^\infty(M)$ (given by the metric g on M and a metric g_E on E),

$$S_2(\phi) = -\langle \phi, Q\phi \rangle$$

for a hermitian operator Q.
(2) I is a *classical interaction functional*, i.e., I lies in $\mathcal{I}_c$, the subspace of local functionals which are at least cubic.

In practice $Q = D + m^2$ for the non-negative Laplacian D and some $m > 0$.

0.0.7. *Data for a quantum theory.* It consists of classical data (M, g, E) and a *quantum action* $S[-]$ which is a family of action functionals $S[L]$ indexed by the scale $L \in (0, \infty)$ and satisfying certain (subtle) properties.

(1) The quadratic part Q is independent of L:

$$S[\Lambda] = S_2 + I[\Lambda] \quad \text{and} \quad S_2(\phi) = -\langle \phi, Q\phi \rangle.$$

So, *quantum action* is a pair $(Q, I[-])$ of a hermitian operator Q and a quantum interaction $I[-]$).
(2) $S[-]$ or equivalently $I[-]$, satisfies the RGE.
(3) $S[-]$ or equivalently $I[-]$, is *asymptotically local* as $L \to 0$.

The last property is subtle, it means that $I[L]$ has an AE $\sum_0^\infty g_i(L)\Phi_i$ for $\Lambda \to 0$, into local functionals $\Phi_i \in \mathcal{O}_l(\mathcal{E})$ rescaled by functions $g_i(L)$ of the scale L.

0.0.8. *The perturbative (i.e., "deformation quantization") framework.* For a quantum theory it means that the quantum interaction $I[-]$ is a formal power series in the parameter $\hbar$ or $T = \hbar/i$. More precisely, each $I[L]$ lies in the space of *quantum interactions* $\mathcal{I}_q = \mathcal{O}_l^+(\mathcal{E})[[\hbar]]$ which is the subspace of $\mathcal{O}_l(\mathcal{E})[[\hbar]]$ given by the requirement that the zero$^{\text{th}}$ term in the $\hbar$-expansion of $I[L]$ is a classical interaction , i.e., $I[L]|_{\hbar=0} \in \mathcal{I}_c$.

In the perturbative framework Costello explains the relation between classical and quantum theories.

0.0.9. *Renormalization group equation (RGE).* It is the requirement that the actions at different scales are compatible – for the observables that are common to two scales the correlations are independent of the scale.

RGE is first stated in terms of the action S and then restated in terms of the interaction I. More importantly, RGE is first stated in terms of the energy scale and then it is reformulated in terms of the scale of the worldline length.

The *energy version* says that for $\Lambda' < \Lambda$, the Λ'-Lagrangian $S[\Lambda']$ is obtained by integrating out in $S[\Lambda]$ the fields on the scale $[\Lambda', \Lambda)$.

$$S[\Lambda'](\phi) \;=\; \hbar \log \big[\int_{\psi \in \mathcal{E}_{[\Lambda',\Lambda)}} \mathcal{D}(\phi+\psi) \;\; e^{\frac{1}{\hbar} S[\Lambda](\phi+\psi)} \big], \qquad \phi \in \mathrm{Ob}_{\leq \Lambda}(M).$$

In terms of $I[\Lambda]$ this means that

$$I[\Lambda'](\phi) \;=\; \hbar \log \big[\int_{\psi \in \mathcal{E}_{[\Lambda',\Lambda)}} \mathcal{D}(\phi+\psi) \;\; e^{\frac{1}{\hbar} (I[\Lambda](\phi+\psi) - \phi Q \phi)} \big], \qquad \phi \in \mathrm{Ob}(M).$$

This RGE is invertible: it applies to arbitrary fields ϕ and it is valid for any Λ', Λ

1. **Intro**

We start with the scalar field theories, i.e., $\mathcal{E} = \mathcal{O}_M$ on a compact manifold M. In the last two sections we extend the results to the case when the fields are sections of a graded vector bundle E and the manifold is not compact.

1.0.10. *Scalar field theories.* Again, we denote by $\mathcal{I}_q$ the space of quantum interactions.

Theorem. **A.** (a) The "forgetting the $\hbar^{n+1}$ term" map

$$\mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)}$$

is a trivial (i.e., "non-empty") torsor for the abelian group of local action functionals $\mathcal{O}_l(\mathcal{E})$.

(b) $\mathcal{T}^{(0)}$ is canonically identified with the space $\mathcal{O}_l^+(\mathcal{E}) \subseteq \mathcal{O}_l(\mathcal{E})$ of local functionals which are at least cubic, i.e., the interacting terms.

Theorem. **B.** A choice of a *renormalization scheme* (RS) splits all torsors $\mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)}$, so it gives an isomorphism of spaces of theories $\mathcal{T}^{(\infty)}$ and the space $\mathcal{I}_q$ of quantum interaction functionals.

1.0.11. *Topics.*

- A version of RG flow.
- F-graphs and integrals.
- Heat kernel low length cutoff ("high energy cutoff").
- The origin of infinities in weight integrals.
- F-graph weights are integrals over maps from the graph to M.
- Precise definition of EQFT and precise statements of the main theorems A and B.
- Renormalization schemes and how they extract the "singular part of weights".
- Local counterterms via a RS.
- Proofs of theorems A and B.
- Extension to vector bundles.

- Extension to noncompact manifolds.

2. RGE for interaction functionals

2.0.12. *Local functionals.* This is the property of action or interaction functionals in a classical theory.

2.0.13. *The data for a perturbative EQFT.* (M, g) is a Riemannian manifold.

An EQFT on M will be given by by the family of actions

$$S[\Lambda] \;=\; S_2 + I[\Lambda]$$

with

- $S_2(\phi) = \; -\langle \phi, Q\phi \rangle$ where
$$Q \;=\; D + m^2$$
for the non-negative Laplacian D and some $m > 0$.
- $\langle -, - \rangle = \langle -, - \rangle_M$ is the inner product on $\mathcal{E}(M) = C^\infty(M)$ given by the metric g on M.
- $I[\Lambda] \in \mathcal{I}_q$.

2.0.14. *RGE for S and I.* For S is says that for $\Lambda' < \Lambda$, the effective Λ'-Lagrangian $S[\Lambda']$ is obtained from by integrating out in $S[\Lambda]$ the fields on the scale $[\Lambda', \Lambda)$.

$$S[\Lambda'](\phi) \;=\; \hbar \log \big[\int_{\psi \in \mathcal{E}_{[\Lambda', \Lambda)}} \mathcal{D}(\phi + \psi) \;\; e^{\frac{1}{\hbar} S[\Lambda](\phi + \psi)} \big], \qquad \phi \in \mathrm{Ob}_{\leq \Lambda}(M).$$

Lemma. (a) This can be restated in terms of $I[\Lambda]$ as

$$I[\Lambda'](\phi) \;=\; \hbar \log \big[\int_{\psi \in \mathcal{E}_{[\Lambda', \Lambda)}} \mathcal{D}(\phi + \psi) \;\; e^{\frac{1}{\hbar} (I[\Lambda](\phi + \psi) - \phi Q \phi)} \big], \qquad \phi \in \mathrm{Ob}(M).$$

(b) This RGE for I is invertible:

(1) The field ϕ is arbitrary.
(2) The equation is valid for any Λ', Λ

2.0.15. *Stochastic $\hbar$-normalization of measures on vector spaces.*

2.0.16. *RGE with one end at $\Lambda = \infty$.* We think of S as $S[\infty]$ (all energies allowed), therefore the RGE equation for $\Lambda < \infty$ should say that

$$I[\Lambda](\phi) \;=\; \hbar \log \big[\int_{\psi \in \mathcal{E}_{[\Lambda, \infty)}} \mathcal{D}(\phi + \psi) \;\; e^{\frac{1}{\hbar} (I[\Lambda](\phi + \psi) - \phi Q \phi)} \big], \qquad \phi \in \mathrm{Ob}(M).$$

However, this integral is ill-defined (for instance it is over an infinite dimensional vector space).

3. **Feynman graphs**

Our goal is to make precise sense of the RG flow, i.e., of the integral that integrates out the fields which are outside of a given scale.

While this integral makes sense when one integrates out a finite dimensional space of fields, we also need an infinite dimensional version in order to go from the classical action S which is roughly $S[\Lambda]$ for energy $\Lambda = \infty$, to the finite scale action $S[\Lambda]$. (Actually, we work with interactions I and $I[\Lambda]$ rater then with actions $S, S[\Lambda]$.)

3.0.17. *Strategy:* **A.** *Operators W_P.* The first part of the strategy is to replace the actual RGF integral with its formal (perturbative) expansion.[(54)] This expansion can be viewed as an operator W which provides an action W_P of propagators $P \in S^2\mathcal{E}$ on the space $\mathcal{I}_q$ of quantum interactions I.

The terms in the expansion are indexed by connected Feynman graphs. We start with defining operators W_P in terms of graphs. Then we notice that when $\mathcal{E}$ is finite dimensional this is really the value of the RGF-type integral and in general this is just a formal expansion of a (formal) RGF integral. of , the inverse P of the operator Q (called propagator) and the interaction I. We will actually

We use the class of stable Feynman graphs to produce an action of the space $S^2\mathcal{U}$ of propagators (on the vector space $\mathcal{U}$), on the space of quantum interactions $\mathcal{I}_q(\mathcal{U}) = \mathcal{O}^+(\mathcal{U})[[\hbar]]$ on $\mathcal{U}$.

3.0.18. *Strategy:* **B.** *Double length cutoffs and RGF propagators $W_{\varepsilon,L}$.* The second step deals with the problem that the propagator P that is natural here, the integral kernel of Q^{-1}, does not fit in the above construction of W_P. The problem is that it is a distributional section of $E\boxtimes E$ on M^2, so it does not lie in the space of propagators $S^2\mathcal{E}\subseteq\mathcal{E}\otimes\mathcal{E}$ which is the space of smooth sections of $E\boxtimes E$.

One replaces the above propagator $P = Q^{-1}$ with its cutoffs which one can define

- In the energy picture: $P_{[\Lambda',\lambda)}$ comes from the spectral decomposition of the operator Q.
- In the length picture: $P_{\varepsilon,L}$ comes from the heat kernel formula $P = \int_0^\infty dl\ K_l$ for the propagator. The cutoff is simply $P_{\varepsilon,L} \stackrel{\text{def}}{=} \int_\varepsilon^L dl\ K_l$.

C. The meaning of the worldline picture. We saw that the length enters through the formula for the propagator in terms of the heat kernel. This is really the formula $\int_0^\infty dl\ e^{-lQ} = Q^{-1}$ in terms of integral kernels K_l, P for e^{-lQ}, Q^{-1}.

xxx

[54] One can hope that this is the asymptotic expansion of the functional integral.

3.1. **Operators W_P on $\mathcal{I}_q$.**

3.1.1. *Stable Feynman graphs.* A *Feynman graph* γ is a graph with

- A subset E_γ of univalent vertices (called *external* vertices or tails).
- A coloring of vertices by *genus* $g; V_\gamma \to \mathbb{N}$.

γ is *stable* if vertices of genus 0 are at lest trivalent.

We denote by V_γ, T_γ, the vertices and terminal (external) vertices. By $\mathcal{E}_\gamma, H_\gamma$ the edges and half-edges where *half-edges* are flags, i.e., incident pairs of an edge and an inner vertex.

Let $b_1(\gamma)$ be the first Betti number of γ. The genus of the graph γ is

$$g_\gamma \;=\; b_1(\gamma) \;+\; \sum_{v\in V_\gamma} g_v.$$

Let k_v be the *valency* of a vertex v and $k_\gamma = |T_\gamma|$ be the *external valency* of a graph γ.

We denote by $s\mathcal{FG}$ the moduli of *stable Feynman graphs*, this is the disjoint union of $B(\mathrm{Aut}(\gamma))$ over all isomorphism classes of graphs γ.

3.1.2. *The weight $w_P^\gamma(I) \in \mathcal{O}^{(|T_\gamma|)}(\mathcal{U})$ of a graph γ with respect to (vector space $\mathcal{U}$,propagator P,interaction I).* We consider a vector space $\mathcal{U}$ equipped with a *propagator*

$$P \;\in\; S^2\mathcal{U}$$

and an interaction $I \in \mathcal{O}^+(\mathcal{U})$.

The idea is that any graph γ specifies an invariant of $(\mathcal{U}, P, I)$ defined as the contraction of two tensors

$$w_P^\gamma(I) \;\stackrel{\mathrm{def}}{=}\; \langle P_\gamma, I_\gamma\rangle \;\in\; \mathcal{O}^{(|T_\gamma|)}(\mathcal{U}).$$

Here,

- $P_\gamma(a) \in \mathcal{U}^{\otimes H_\gamma}$ is obtained by labeling edges and then tensoring at vertices. The internal edges are labeled by the propagator P and the external ones by a variable vector $a \in \mathcal{U}$. (Notice that $P \in \mathcal{U}\otimes\mathcal{U}$ contributes one $\mathcal{U}$ per each internal half-edge and a contributes one $\mathcal{U}$ per external half-edge.)
- $I_\gamma(a) \in (\mathcal{U}^{\otimes H_\gamma})^{\mathfrak{v}}$ is obtained by labeling inner vertices and then tensoring along edges. Here, internal vertex v is labeled by I_{g_v,k_v}. (So the degree of I_γ is $\sum_{v\in V_\gamma} \deg(I_{g_v,k_v} = \sum_{v\in V_\gamma} k_v$, the sum of valencies of inner vertices, i.e., exactly the number of half-edges.)

So, $[w_P^\gamma(I)](a) \stackrel{\mathrm{def}}{=} \langle P_\gamma(a), I_\gamma\rangle$ is a homogeneous function $w_P^\gamma(I)$ of $a \in \mathcal{U}$. Its degree is the number of places where a appears in P_γ, i.e., the number of external edges.

3.1.3. *The action W_P of a propagator $P \in S^2\mathcal{U}$ on the space of quantum interactions $\mathcal{I}_q = \mathcal{O}^+(\mathcal{U})[[\hbar]]$.* Here we use a propagator $P \in S^2\mathcal{U}$ to act on the space of quantum interactions. The result $W_P I$ has the *free energy* form, i.e., it is the sum over all connected configurations, i.e., connected Feynman graphs[(55)]

$$W_P I \;=\; \boxed{W(P,I)} \;\stackrel{\text{def}}{=}\; \int_{\gamma\in\ cs\mathcal{FG}} \hbar^{g_\gamma}\; w_P^\gamma(I).$$

The operator W_P has expansion

$$W_P = \sum_{i,k\geq 0} \hbar^i\; W_{P,i.k}$$

in the sense of $W_{P,i.k}I \stackrel{\text{def}}{=} (W_P I)_{ik}$. We denote by $s\mathcal{FG}_{ik}$ the submoduli of graphs γ with genus $g_\gamma = i$ and external valency $k_\gamma = k$, then

$$W_{P,i,k}I \;=\; \int_{\gamma\in\ cs\mathcal{FG}_{ik}} w_P^\gamma(I).$$

Lemma. Series $W_P I$ converges.?

Proof. The reason is that we imposed stability condition on graphs!

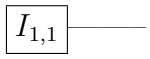
yyy

3.1.4. *Examples of W_{ik}'s.* Since $W_{i,k}$ is an integral over $cs\mathcal{FG}_{i,k}$, the point is to describe the moduli $cs\mathcal{FG}_{i,k}$,

$\boldsymbol{cs\mathcal{FG}_{0,3}}$. These are the connected graphs with genus zero and three external vertices. $g_\gamma = 0$ means that $b_1(\gamma) = 0$ (γ is a tree) and each internal vertex v has genus 0. This implies that internal vertices have valency ≥ 3, and since γ is a tree this means precisely one internal vertex. So, $cs\mathcal{FG}_{0,3}$ has one point – the D_4 graph γ with $\mathrm{Aut}(\gamma) \cong S_3$.

$\boldsymbol{cs\mathcal{FG}_{0,4}}$. Here the number of internal vertices is ≤ 1, and the moduli has two points, α has one internal vertex which is 4-valent, i.e., 4 external spokes are protruding from it. β has two internal vertices, they are joined by a single internal edge and both are 3-valent.

$\boldsymbol{cs\mathcal{FG}_{1,1}}$. If $b_1 = 0$ then γ is a tree with one external vertex. An internal vertex v can not have zero genus – since γ is a tree this would imply that there are at least 3 external vertices. So, $g_\gamma = 1$ implies that there is precisely one internal vertex and it has genus 1. The picture (with the internal vertex v boxed and labeled by $I_{g_v,k_v} = I_{1.1}$)



If $b_1 = 1$ then γ has one loop and internal vertices have zero genus. This allows one internal vertex with valency 3 – two of the form the loop and the third is external.

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[55] $s\mathcal{FG}$ is the free abelian semigroup on $cs\mathcal{FG}$, so $\int_{s\mathcal{FG}} = e^{\int_{cs\mathcal{FG}}}$.

The formula for new I has the free energy form since interaction I appears as the logarithm of the amplitude e^I.

3.1.5. *Interpretation of $W_P I$ as the free energy perturbative expansion of a functional integral.* This works when $\mathcal{U}$ is finite dimensional (unreasonable) and P is invertible (reasonable).

Lemma. (a) If $\mathcal{U}$ is finite dimensional and P is invertible, for any $a \in \mathcal{U}$[56]

$$(W_P I)(a) \;=\; \hbar \log \Big[\int_{\mathcal{U}} dx \; e^{\frac{1}{\hbar}[I(a+x) - P^{-1}(x,x)]} \Big].$$

Proof. This is a Wick lemma computation.

Remark. The point is that the the operator W_P generalizes the above functional integral to infinite dimensional spaces where Wick lemma is not an equality but only an asymptotic expansion.

3.1.6. *Functional integral in terms of ∂_P.* ∂_P is just the constant differential operator given by $P \in S\mathcal{U}$, for $P = \sum \; P' \otimes P''$ this means $\partial_P = \sum \; \partial_{P'} \partial P''$.

Lemma.

$$W_P I \;=\; \hbar \log \Big[\int_{\mathcal{U}} dx \; e^{\hbar \partial_P} e^{\frac{1}{\hbar} I(a+x)} \Big].$$

Proof. Just a repackaging of the preceding formula.

Corollary. W is an action of the vector space $S^2\mathcal{U}$ on $\mathcal{I}_q$:

$$W_Q \circ W_p \;=\; W_{P+Q}.$$

3.1.7. *The functional analysis for the infinite dimensional setting $\mathcal{U} = \mathcal{E}$.* We work with in the tensor category of *nuclear Frechet spaces* with *completed projective* tensor product.

(1) $\mathcal{E}$ is a nuclear Frechet space.
(2) $\mathcal{O}(\mathcal{E}) \stackrel{\text{def}}{=} \; \prod_{n \geq 0} \; \mathcal{O}^{(n)}(\mathcal{E})$, for

$$\mathcal{O}^{(n)}(\mathcal{E}) \;=\; S^n(\mathcal{E}^\vee) \;=\; [(\mathcal{E}^{\otimes} n)^\vee]^{S_n},$$

where the dual is the *continuous dual.*
This is a (topological?) algebra.
(3) **Problem.** We need the propagator P to be in $\mathcal{E} \otimes \mathcal{E}$.
(4) Then, everything works as above, with $P_\gamma \in \; \mathcal{E}^{\otimes H_\gamma}$ and $I_\gamma \in \; (\mathcal{E}^{\otimes H_\gamma})^\vee$.

[56] Recall the normalization of the Haar measure on $\mathcal{U}$.

3.1.8. *An explanation of singularities in QFT.* The above scheme fails when the propagator P is chosen as the integral kernel for Q^{-1}. Since P is a distributional section of $E\boxtimes E$ on M^2, it does not live in the correct tensor product $\mathcal{E}\otimes\mathcal{E}$ = smooth sections of $E\boxtimes E$.

So, one can say that the infinities in QFT arise because Q^{-1} does not lie in the correct tensor product.

Question. Is the problem that one should change the tensor structure? Is this accomplished by FAs?

4. **Sharp and smooth cutoffs**

The RGF will be given by operators W_P for well chosen propagators P.

We consider the integral kernels

- P for Q^{-1} and
- K_l for e^{-lQ}.

We can not quite use P as the propagator since it is not smooth.

A. RGF flow in the energy picture.

4.0.9. *Laplacian eigenvectors.* On a compact M, the Laplacian has isolated spectrum with eigenvectors

$$De_i \;=\; \lambda_i e_i.$$

For $Q = D+m^2$ this means $Qe_i \;=\; (\lambda_i+m^2)e_i$, hence $Q^{-1}e_i \;=\; (\lambda_i+m^2)^{-1}e_i$, Therefore, the integral kernel P for Q^{-1} takes form in $L^2(M^2)$

$$P \;=\; \sum \frac{1}{\lambda_i+m^2}\; e_j\otimes e_j.$$

(Really, $(\sum\; c_j\; e_j\otimes e_j)e_i = \;\sum\; c_j\; e_j\cdot\langle e_j,e_i\rangle = c_j e_j$.)

4.0.10. *Energy cutoffs of the propagator.* The U-cutoff of P, for $U\subseteq\mathbb{R}^{>0}$ is

$$P_U \;\stackrel{\text{def}}{=}\; \sum_{\lambda_j\in U} \frac{1}{\lambda_i+m^2}\; e_j\otimes e_j$$

Lemma. If $U\subseteq\mathbb{R}^{>0}$ is bounded above, then P_U is a smooth function on M^2.

4.0.11. *RGF flow $W_{P_{[\Lambda',\Lambda)}}$.* Now we can replace the formal expansion of a formal integral

$$(W_P I)(a) \;=\; \hbar \log\big[\; \int_{\phi\in\mathcal{E}} \mathcal{D}\phi\; e^{\frac{1}{\hbar}[I(a+\phi)-\phi Q\phi]}\big],$$

with actual identities for $\Lambda' < \Lambda$

$$(W_{P_{[\Lambda',\Lambda)}} I)(a) \;=\; \hbar \log\big[\; \int_{\phi\in\mathcal{E}_{[\Lambda',\Lambda)}} e^{\frac{1}{\hbar}[I(a+\phi)-\phi Q\phi]}\big].$$

B. RGF flow in the wordline length picture. We will write the propagator $P = Q^{-1}$ as an integral $P = \int_0^\infty \; dl\; K_l$ of the heat kernel K_l for Q. This suggests the cutoffs of the propagator $P_{\varepsilon,L} \stackrel{\text{def}}{=} \int_\varepsilon^L \; dl\; K_l$.

4.0.12. *Heat kernel K for Q.* The standard heat kernel is the integral kernel K_l^0 for e^{-lD}, the one for $e{-}lQ = e^{-lm^2}e^{-lD}$ is $K_l = e^{-lm^2}K_l^0$. It is related to the integral kernel$P$ for Q^{-1} by

Lemma. $\int_0^\infty \; dl\; K_l \;=\; P$.

Proof. $\int_0^\infty \; dl\; e^{-lQ} = \; [\frac{e^{-lQ}}{-Q}]_0^\infty = \; Q^{-1} = \; P$.

4.0.13. *The double length cutoffs $P_{\varepsilon,L}$ of the propagator P and the length-RGF $W_{\varepsilon,L}$.* For $0 \le \varepsilon < l \le \infty$ define

$$P_{\varepsilon,L} \stackrel{\text{def}}{=} \int_\varepsilon^L \; dl\; K_l.$$

Terminology:

- *ultraviolet* = high energy = short length $= \varepsilon$.
- *infrared* = low energy = long length $= L$.

Lemma. For $\varepsilon > 0$and $L \le \infty$, $P_{\varepsilon,L}$ is a smooth section, so

$$W_{\varepsilon,L} \;\stackrel{\text{def}}{=}\; W_{P_{\varepsilon,L}}$$

is well defined.

Proof. It follows from the formula

$$P_{\varepsilon,L} = \; \sum \; \frac{e^{-(\lambda_i+m^2)L} - e^{-(\lambda_i+m^2)\varepsilon}}{\lambda_i + m^2}\; e_i{\otimes}e_i$$

that is gotten from the expansion of the heat kernel:

$$P_{\varepsilon,L} = \; \int_\varepsilon^L \; dl\; K_l = \; \int_\varepsilon^L \; \sum \; e^{-l(\lambda_i+m^2)}\; e_i{\otimes}e_i = \; \sum \; \big(\int_\varepsilon^L \; e^{-l(\lambda_i+m^2)}\big) \; e_i{\otimes}e_i.$$

Remarks. Now we can define the length RGF as the family of operators $W_{\varepsilon,L}$.

4.0.14. *Graphical view (Feynman graphs) on the RGF operators $W_{\varepsilon,L}$.*

4.0.15. *Infinitesimal RGF.*

5. **Geometric interpretation of Feynman graphs**

5.0.16. *The role of Feynman graphs from the points of view of: (i)* functional integrals*, and (ii)* worldlines. point of view, the perturbative expansion

For functional integrals, Feynman graphs appear as an organizational tool – the index set of the summands in the perturbative expansion.

However, from the *worldline* point of view, the perturbative expansion of a functional integrals describes a new QFT whose fields are trajectories of systems of particles in M and these are described as maps from Feynman graphs with proper ("internal") time. Here, Feynman graphs appear as topological types of trajectories, i.e., descriptions of all possible interactions of a system of particles.

Remarks. (0) The worldline view appears here from the perturbative picture, as an *asymptotic picture* of the QFT in the original functional integral.

(1) Notice that the String Theory is also an asymptotic theory! Moreover, it seems to be the direct extension of the worldline view, obtained by upgrading Feynman graphs to surfaces.

Question. Certain elements of the theory of surfaces appear in the Feynman graph formalism – the genus of a vertex and the notion of stable graphs which corresponds to stable surfaces. Why?

5.0.17. *The worldline view on propagators.* The propagator P appears from several points of view as

(1) Correlator of values of fields for the free theory given by Q

$$P(x,y) \;=\; \langle \mathcal{O}_x, \mathcal{O}_y \rangle \;\stackrel{\text{def}}{=}\; \int_{\phi\in\mathcal{E}} \mathcal{D}\phi \; e^{-\int_M \; \phi Q \phi} \; \phi(x)\phi(y).$$

(2) The integral kernel for Q^{-1}.
(3) An integral of the heat kernel for Q

$$P \;=\; \int_0^\infty dl \; K_l.$$

(4) An integral over the space $\mathcal{P}(x,y)$ over all paths in M from x to y. The paths we consider have *proper time*[57], so $\mathcal{P}(x,y) = \sqcup_{\tau\geq 0} \; \mathcal{P}_\tau(x,y)$ where index τ means

[57]It is not related to the time in the spacetime M.

paths that take time τ from x to y, i.e., $f : [0, \tau] \to M$ and $f(0) = x$, $f(\tau) = y$. So,

$$P \;=\; \int_{\tau=0}^{\infty} d\tau \;\; \int_{f\in\mathcal{P}_\tau(x,y)} \mathcal{D}\mathcal{W}(f) \;\; 1$$

where $\mathcal{W}$ is the Wiener measure on the space of paths.

(5) The partition function of the QFT where the space of fields is $\mathcal{P}(x, y)$ and the action is the kinetic energy

$$E(f) \;=\; \int_0^\tau dt\ |df|^2.$$

This is of course an intuitive description and we write it as an intuitive functional integral

$$P \;=\; \int_{f\in\mathcal{P}(x,y)} e^{-E(f)}.$$

Also, the heat kernel K_l appears as

- (i) The integral kernel for e^{-lQ}.
- (ii) The probability for the particle $f : [0, \tau] \to M$ to be at x when $t = 0$ and at y when $t = \tau$.

Here (4) and (ii) are the worldline views on the propagator P and on K_l.

5.0.18. *Some relations between different approaches to P.* The philosophical interpretations (1) and (5) are of no immediate importance. The precise meaning of (5) is the formula (4), since (4) is obtained by making sense of the functional integral (5) through the Wiener measure.

(3) is equivalent to (2) when we remember (i) (K_l is the integral kernel for e^{-lQ}).

The worldline view (4) on P is obtained from (3) by using the worldline view (ii) on the heat kernel K_l.

(4) can also be obtained from (1) by the Wick lemma calculation of the formal expansion of the functional integral (1).

5.1. **The worldline view on the correlation functions $\langle \mathcal{O}_{x_1}, ..., \mathcal{O}_{x_n}\rangle$.** For $S = I - \phi Q\phi$, we define formally correlation function as a function integral

$$\mathbb{E}(x_1, ..., x_n)_{formal} \stackrel{\text{def}}{=} \langle \mathcal{O}_{x_1}, ..., \mathcal{O}_{x_n}\rangle_{formal} \stackrel{\text{def}}{=} \int_{\phi\in\mathcal{E}} e^{\frac{1}{\hbar}S(\phi)}\phi(x_1)\cdot\cdot\cdot\phi(x_n).$$

We define the actual correlator $\mathbb{E}(x_1, ..., x_n)$ as the perturbative expansion of the formal correlator $\mathbb{E}(x_1, ..., x_n)_{formal}$.

For $\gamma \in s\mathcal{FG}$ we denote by $Met(\gamma)$ the space of "metrics" on γ, meaning all possible lists of lengths of edges $g : \mathcal{E}_\gamma \to \mathbb{R}_{\geq 0}$. Let $Ms\mathcal{FG} = \int_{\gamma\in s\mathcal{FG}} Met_\gamma$ be the moduli of metrized Feynman graphs.

A metrized graph (γ, g) can be viewed as consisting of edges e parametrized by the corresponding intervals $[0, g(e)]$. Then a "γ-path" $f : \gamma \to M$ is a system of maps $f_e : [0.g(e)] \to M$ and we define its energy as the sum $E(f) = \sum_e E(f_e) = \sum_e \int_0^{g(\gamma)} dt\, |df_e|^2$ of energies of its parts f_e.

Lemma.

$$\mathbb{E}(x_1, ..., x_n) \;=\; \int_{\gamma \in s\mathcal{FG}} \hbar^{-\chi(\gamma)} \quad \int_{Met(\gamma)} \quad \int_{f:\gamma \to M} \quad e^{-E(f)}.$$

Proof. Wick lemma.

Remark. We see again that the passage from the (usually formal, i.e., symbolic) functional integral to its asymptotic expansion is

- Achieved by applying (formally) the Wick lemma.
- Constitutes a passage to the *worldline* view on the subject.

5.1.1. *The double length-cutoff of correlators.* These are obtained by replacing the moduli of metrized Feynman graphs $Ms\mathcal{FG}$ by the submoduli $Ms\mathcal{FG}_{\varepsilon,L}$ of graphs each of whose legs e has length in $[\varepsilon, L]$.

5.2. **Definition of correlators from the effective interaction $I[-]$.**

5.2.1. *Correlators $\langle O_1, .., O_n\rangle$ and RGE.* Here, we are interested in correlators for just one purpose – the formulation of the RGE. The point is that RGE is just a restatement of the requirement that for any observables $O_i \in \mathcal{O}_{(\varepsilon,L]}$, i.e., on the scale $(\varepsilon, L]$, the correlator $\langle O_1, .., O_n\rangle_{\varepsilon',L'}$ calculated at the scale (ε', L') is independent of the choice of the scale (ε', L') (as long as we use a wider scale, i.e., $(\varepsilon', L') \supseteq (\varepsilon, L)$).

More precisely, in the scaled (effective) QFT formalism, an observable O is not uniform at all scales, rather it has incarnations $O_{\varepsilon,L}$ on any scale (ε, L). We are most interested in the case

$$O[L] \;\stackrel{\text{def}}{=}\; O_{0,L}.$$

Now, the *scaled RGE requirement* is that $\langle (O_1)_{\varepsilon,L}, .., (O_n)_{\varepsilon,L}\rangle_{\varepsilon,L}$ is independent of the scale (ε, L).

5.2.2. *For observables in EQFT see [CG].* Observables do not really figure in Costello's book, instead they are treated in the [CG]-paper. This treatment is the theory of *factorization algebras*. It includes

- Two notions of scaled observables: *strict* and *homotopy* observables.
  The difference is that the strict ones satisfy RGE equation strictly and the homotopy ones satisfy RGE on the homotopy level.

- Both form factorization algebras which are quantizations of Poisson factorization algebras of classical observables. However, the homotopy observables form a formally better object since they have the full structure that one wants from the quantization of the algebra of classical observables – the structure of BD factorization algebra.

5.2.3. *Correlators of point observables $\langle O_1, .., O_n\rangle$ form a single correlator distribution $\mathbb{E}^n$ on M^n.* The idea is that for functions f_i on M

$$\mathbb{E}(f_1, ..., f_n) \;=\; \int_{x_\bullet \in M^n} \langle (O_{x_1}, ..., O_{x_n}\rangle \cdot f_1(x_1)\cdot\cdot f_n(x_n).$$

It contains the information of all correlators of observables supported at points (Here D_i are differential operators.)

$$\langle O_{D_1 x_1}, ..., O_{D_n x_n}\rangle.$$

5.2.4. *The graphs for correlators.*

Remark. The scale L appears in two ways in the construction of correlators, in the scaled interaction $I[L]$ and the scaled propagator $P[L]$.

Lemma. RGE for $I[L]$'s is equivalent to the independence of the correlators $\mathbb{E}^n_{L,I[L]} \in \mathcal{D}^n(M)$ on the parameter L.

5.2.5. *The meaning of the worldline view on correlators.* The integral over $Ms\mathcal{FG}_{\varepsilon,L}$ means that we consider the evolutions f of a system of particles which interact according to $I[L]$ but are constrained to travel for the proper (internal) time at least L between any two interactions.

6. **Effective QFT: the definition**

6.0.6. *The scope of the theory.* For a moment we consider the massive scalar theories on a compact M, so $\mathcal{E} = C^\infty(M)$ and $Q = D + m^2$.

6.0.7. An effective QFT (EQFT) is a family of quantum interactions $I[L] \in \mathcal{I}_q$, indexed by the length scale $L \in (0, \infty)$, such that

(1) Renormalization group equation (RGE)

$$W_{\varepsilon,L} I[\varepsilon] \;=\; I[L].$$

(2) For any $i, k \in \mathbb{N}$, the component $I_{ik}[L] \in S^k(\mathcal{E}^\vee)$ has an asymptotic expansion for small L

$$I_{i,k}[L] \overset{AE}{=} \sum_{r=0}^{\infty} g_r(L)\ \Phi_r,$$

into local functionals $\Phi_r \in \mathcal{O}_l(\mathcal{E})$ of degree k, and with coefficients functions $g_r \in C^\infty(\mathbb{R}^{>0}{}_L)$ of the scale L.[58]

6.0.8. *Quantization theorems.* Notice that the definition of an effective action makes sense when one relates the ring $\Bbbk[[\hbar]]$ with $\Bbbk[[\hbar]]/\hbar^{n+1} = \Bbbk[\hbar]/\hbar^{n+1}$. For $0 \leq n \leq \infty$ let $\mathcal{T}^{(n)}$ be the space of effective Quantum Field theories for $\Bbbk[[\hbar]]/\hbar^{n+1}$, i.e., *defined modulo* $\hbar^{n+1}$.

Theorem. **AB.** Fix S_2.[59]

(a) The "forgetting the $\hbar^{n+1}$ term" map

$$\mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)}$$

is a trivial (i.e., "non-empty") torsor for the abelian group of local action functionals $\mathcal{O}_l(\mathcal{E})$.

(b) $\mathcal{T}^{(0)}$ is canonically identified with the space $\mathcal{I}_c = \mathcal{O}_l^+(\mathcal{E})$ of classical interactions.

(c) A choice of a *renormalization scheme* (RS) Σ splits all torsors $\mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)}$, so it gives an isomorphism (the effective σ-quantization isomorphism) of spaces of theories and of quantum interactions:

$$EQ^\Sigma :\ \mathcal{I}_q \overset{\cong}{\to} \mathcal{T}^{(\infty)}$$

which is compatible with (b).

Remark. The isomorphism EQ^Σ is unnatural in the sense that it requires a choice of Σ.

Question. Do we know any natural choice of Σ?

---

[58] This is an *asymptotic expansion* (AE) in the sense that there exists a nondecreasing sequence $d_r \to +\infty$ such that for each field $\phi \in \mathcal{E}$

$$\lim_{\varepsilon \to 0} \frac{I_{ik}[L](\phi) - \sum_{\rho=0^r}\ g_\rho(L)\Phi_\rho(\phi)}{L^{d_r}} \ =\ 0.$$

So, this is an asymptotic expansion in the topological vector space

$$S^k(\mathcal{E}^\vee) = \ [(\mathcal{E}^{\otimes k})^\vee]^{S_k}$$

for the weak topology given by the fields $\phi \in \mathcal{E}$.

[59] At this stage the book considers $\mathcal{E} = C^\infty(M)$ and $S_2 = \phi Q \phi$ for $Q = D + m^2$.

6.0.9. *About the proof.* (b) is clear by definition of $\mathcal{T}^{(0)}$.

We first prove the "less natural" version (c). Then (a) follows "easily" since (c) contains the existence part of the claim (a).

For (c) we need to pass from a quantum interaction I to an effective quantum interaction $I^{\Sigma}[-]$. Since I is an attempt to have an action adequate for all (length) scales, it should fit with $I[-]$ as a version of $I[0]$. So, we would like to transition between the two by

$$I[L] \;=\; W_{L,0}I \;=\; \lim_{\varepsilon\to 0}\; W_{\varepsilon,L}I \quad \text{and} \quad I = \lim_{\varepsilon\to 0}\; I(\varepsilon).$$

The limits do not exist, rather, in the formula $I[L] = \lim_{\varepsilon\to 0}\; W_{\varepsilon,L}I$ we need to replace I with its Σ-regularized version $I - I^{CT,\Sigma}(\varepsilon)$. The *counterterm* $I^{CT,\Sigma}$ is defined uniquely by Σ and the requirement that for each L the limit in the following definition exists:

$$I[L] \;\stackrel{\text{def}}{=}\; \lim_{\varepsilon\to 0}\; W_{\varepsilon,L}I - I^{\mathrm{C}}(\varepsilon).$$

In the converse direction, again I is a renormalized version of the limit $\lim_{\varepsilon\to 0}\; I(\varepsilon)$.

Question. Is $\mathcal{T}^{(\infty)}$ naturally a torsor over $\mathcal{T}^{(0)} = \mathcal{I}_c$? (The group in question would be the Kontsevich motivic group??)

Remark. The heart of the construction is the handling of the singularities that arise in $W_{\varepsilon,L}$, i.e., in the graph weights $w^{\gamma}_{\varepsilon,L}$, as $\varepsilon \to 0$, i.e., as we allow arbitrarily high energies. This extraction of singularities is achieved through the science of counterterms.

7. Extracting the singular part of the weights of geometric interpretation of Feynman graphs

7.0.10. *The algebra $\mathcal{P}$ of period functions.* One says that a complex number α is a *period* if it has a presentation

$$\alpha \;=\; \int_{\gamma}\; \omega$$

as an integral of a form ω of top degree, over a middle dimensional cycle γ, with all data defined over $\mathbb{Q}$:

- complex algebraic variety X with a normal crossing divisor D,
- a form $\omega \in \Omega^{\dim(X)}(X)$ and
- a relative homology class

$$\gamma \;\in\; H_{\dim(X)}[X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q}].$$

Remark. [Kontsevich-Zagier] "all" constants in mathematics should be periods.

One says that a function α on $(0,\infty)$ is a *period function* if it has a presentation as a family of periods

$$\alpha(\varepsilon) \;=\; \int_\gamma \omega_\varepsilon$$

with the data defined over $\mathbb{R}$:

- a complex algebraic variety X with a normal crossing divisor D,
- A Zariski open subset $U\subseteq\mathbb{A}^1$ which is defined over $\mathbb{R}$ and its real points $U(\mathbb{R})]\subseteq\mathbb{A}^1(\mathbb{R})=\mathbb{R}$ contain $\mathbb{R}^{>0}$.
- a smooth map of a pair (X,D) into U, such that $(X(\mathbb{C}),D(\mathbb{C}))$ is locally trivial in the C^∞-category.
- a relative form $\omega\in\Omega^{\dim(X)}(X/U)$ and
- a relative homology class in the fiber at $1\in U$

$$\gamma \;\in\; H_{\dim(X)}[X_1(\mathbb{C}),D_1(\mathbb{C});\mathbb{R}].$$

Remark. The choice of U was needed, i.e., $U=\mathbb{A}^1$ (does not suffice) so that we could require that (X,D) is "nice" over U. The condition that $(X(\mathbb{C}),D(\mathbb{C}))$ is locally trivial in the C^∞-category is needed just to be able to transport the class γ from the special fiber at 1 to any fiber. Notice that while γ_ε may be multivalued for general $\varepsilon\in U$ (monodromy of periods), there is a canonical choice for $\varepsilon>0$ (the extension along $\mathbb{R}^{>0}$).

7.0.11. *The singularities of weights $w^\gamma_{\varepsilon L}I$ of a Feynman graph γ.* Our control of these singularities is the heart of the construction of the counterterms. It is contained in the following theorem which provides a *double asymptotic expansion* for the weights $w^\gamma_{\varepsilon L}I$ (first in ε and then in L). This is the one "hard" ingredient of the construction of the effective actions. The proof is based on a parallel understanding of the asymptotic expansion of heat kernels in the Berline-Getzler-Vergne theory.

For any choice of γ and I, $(w^\gamma_{\varepsilon L}I)(\phi)$ depends on the field ϕ and $\varepsilon,L\in\mathbb{R}^{>0}$. We will view it as a function of ε with values in a nuclear space

$$w^\gamma_{\varepsilon,-}I \;\in\; \mathcal{O}(\mathcal{E},\; C^\infty(\mathbb{R}^{>0}_L)) \;\cong\; \mathcal{O}(\mathcal{E})\otimes C^\infty(\mathbb{R}^{>0}_L).$$

Theorem. For any $I\in\mathcal{I}_q$ the integral $w^\gamma_{\varepsilon,L}I$ has an AE for small ε in the the nuclear space $\mathcal{O}(\mathcal{E})\otimes C^\infty(\mathbb{R}^{>0}_L)$,

$$w^\gamma_{\varepsilon,L}I \overset{AE}{=} \sum_0^\infty g_i(L)\;\Phi_i(L,-)$$

such that

(1) the coefficient functions are period functions: $g_i\in\mathcal{P}$; with only finite order poles at 0;

(2) each $\Phi_i(L,-) \in \mathcal{O}(\mathcal{E}) \otimes C^\infty(\mathbb{R}_L^{>0})$ has an AE in the nuclear space of local functionals into local functionals, for small $\mathbb{L}$,

$$\Phi_i(L,-) \;=\; \sum_{j=0}^{\infty} f_{ij}(L)\ \Phi_{ij}$$

such that the coefficient functions are smooth $f_{ik} \in C^\infty(\mathbb{R}_L^{>0})$ (and $\Phi_{ij} \in \mathcal{O}_l(\mathcal{E})$).

7.0.12. *Renormalization schemes.* Let $\mathcal{P} = \mathcal{P}(0,\infty)$ be the subring of $C^\infty(0,\infty)$ generated by all period functions.

A *renormalization* scheme is a choice of a complementary subspace $\Sigma = \mathcal{P}_{<0}$ to the subalgebra $\mathbb{P}_{\geq 0} \subseteq \mathcal{P}$. The projections

$$\Sigma = \mathcal{P}_{<0} \xleftarrow{Sing^\Sigma} \mathcal{P} \xrightarrow{Reg^\Sigma} \mathcal{P}_{\geq 0}$$

are called the Σ-*singular* part and the Σ-*regular* part.

Lemma. These constructions extend to functions which have asymptotic expansions for small ε

$$W(\varepsilon) \stackrel{AE}{=} \sum_{0}^{\infty} g_i(\varepsilon)\Phi$$

such that the coefficient functions g_i are periods.

Proof. First we define

$$Sing^\Sigma W \stackrel{\text{def}}{=} \lim_{r\to\infty} Sing^\Sigma(\sum_{0}^{r} g_i\Phi) \stackrel{\text{def}}{=} \lim_{r\to\infty} \sum_{0}^{r} Sing^\Sigma(g_i)\Phi$$

and then $Reg^\Sigma(I)$ is defined as $1 - Sing^\Sigma$. This is based on

7.0.13. *Sublemma.* If W has asymptotic expansions for small ε

$$W(\varepsilon) \stackrel{AE}{=} \sum_{0}^{\infty} g_i(\varepsilon)\Phi$$

then for $i >> 0$ the coefficients g_i have zero limit at $\varepsilon = 0$. (What is needed is $i > 0$ and $d_{i-1} > 0$.)

Proof. The limit of

$$-\frac{g_i(\varepsilon)}{\varepsilon^{d_{i-1}}} = \frac{W(\varepsilon) - \sum_{j=0}^{i}\ g_j(\varepsilon) - W(\varepsilon) - \sum_{j=0}^{i-1}\ g_j(\varepsilon)}{\varepsilon^{d_{i-1}}}$$

$$= \ \varepsilon^{d_i - d_{i-1}} \frac{W(\varepsilon) - \sum_{j=0}^{i}\ g_j(\varepsilon)}{\varepsilon^{d_i}} - \frac{W(\varepsilon) - \sum_{j=0}^{i-1}\ g_j(\varepsilon)}{\varepsilon^{d_{i-1}}}$$

is zero. So, a soon as $d_{i-1} > 0$ we have $\lim_{\varepsilon\to 0}\ g_i(\varepsilon) = \ 0$.

Remark. The choices of a complement to a subspace $U \subseteq V$ form a torsor for $\mathrm{Hom}(V/U, U)$ since any choice of a complement Σ gives a parametrization of complements by

$$\mathrm{Hom}(V/U, U) \cong \mathrm{Hom}(\Sigma, U) \ni A \mapsto \Gamma_A \stackrel{\mathrm{def}}{=} (id + A)\Sigma.$$

So, renormalization schemes form a torsor for $\mathrm{Hom}(\mathcal{P}/\mathcal{P}_{\leq 0}, \mathcal{P}_{\leq 0})$.

Remark. Physicists use a number of renormalization schemes which do not fall under this formalism (they do not use the space of periods), for instance the "minimal subtraction scheme". These schemes are mechanisms for extending moderately singular functions across singularity as distributions. Though the notion of a RS (a characterization of admissible ways of extending across singularities) is not formalized in physics, there is an understanding of which mechanisms are meaningful. (The approach via periods is due to Kontsevich?)

7.0.14. *The Σ-regular part $Reg^{\Sigma}(I)$ of I.* We make a choice of a RS Σ. Since in the asymptotic expansion $w^{\gamma}_{\varepsilon L} I \stackrel{AE}{=} \sum_0^{\infty} g_i(\varepsilon)\Phi_i$, the coefficient functions g_i are periods, we have a decomposition of $w^{\gamma}_{\varepsilon L} I$ into its singular and regular parts with respect to Σ, say

$$Sing^{\Sigma}[w^{\gamma}_{\varepsilon L} I] = \sum_0^n Sing^{\Sigma}(g_i)(\varepsilon)\Phi_i$$

for sufficiently large n.

Let us summarize the properties of this construction.

Proposition. Let Σ be a renormalization scheme. For any $I \in \mathcal{I}_q$ the Σ-singular part of the integral $w^{\gamma}_{\varepsilon, L} I$ is finite sum

$$Sing^{\Sigma}[w^{\gamma}_{\varepsilon L} I] \ = \ \sum_0^{\infty} f_i(\varepsilon)\Phi_i(L, -).$$

Here $f_i \in \Sigma = \mathcal{P}_{<0}$, while each $\Phi_i(L, -) \in \mathcal{O}_l(\mathcal{E}, C^{\infty}(\mathbb{R}^{>0}_L))$ has an AE for small $\mathbb{L}$, in the nuclear space of local functionals

$$\Phi_i(L, -) \ = \ \sum_{j=0}^{\infty} f_{ij}(L) \ \Phi_{ij}$$

such that the coefficient functions are smooth $f_{ik} \in C^{\infty}(\mathbb{R}^{>0}_L)$ (and $\Phi_{ij} \in \mathcal{O}_l(\mathcal{E})$).

Corollary. The Σ-regular part $Reg^{\Sigma}(w^{\gamma}_{\varepsilon L} I)$ has a limit for $\varepsilon \to 0$ in the nuclear space $\mathcal{O}(\mathcal{E}, C^{\infty}(\mathbb{R}^{>0}_L)) = \mathcal{O}(\mathcal{E}) \otimes C^{\infty}(\mathbb{R}^{>0}_L)$.

Proof. ??????

8. **Construction of local counterterms**

We choose a renormalization scheme $\Sigma = \mathcal{P}_{<0}$.

Theorem. For each interaction I there exists a unique "local counterterm"

$$I^{\mathrm{C}} : \mathbb{R}_{\varepsilon}^{>0} \longrightarrow \mathcal{I}_q \subseteq \mathcal{O}_l(\mathcal{E})[[\hbar]]$$

such that

(1) for each $L > 0$ the limit

$$I^{\Sigma}[L] \stackrel{\mathrm{def}}{=} \lim_{\varepsilon \to 0} W_{\varepsilon,L}(I - I^{\mathrm{C}})$$

exists in $\mathcal{I}_q$.

(2) In the expansion $I^{\mathrm{C}} = \sum_{i.k} \hbar^i I^{\mathrm{C}}_{ik}$ the term $I^{\mathrm{C}}_{ik}(\varepsilon)$ lies in the algebraic tensor product $\mathcal{O}_l(\mathcal{E}) \otimes_{alg} \Sigma$.

Remark. Traditionally one constructs counterterms in terms of Feynman diagrams. Costello has a found a formal proof but he also states the Feynman diagrams proof for the additional intuition that it provides.

8.0.15. *Construction via Feynman diagrams.* Recall that the operator $W = W_{\varepsilon,L}$ is a formal series $W = \sum_{ik} \hbar^i W_{ik}$ with

$$W_{ik} = \int_{\gamma \in s\mathcal{FG}_{ik}} w^{\gamma}_{\varepsilon L}$$

where $s\mathcal{FG}_{ik} = \boxed{\Gamma_{ik}}$ is the moduli of stable Feynman graphs of genus i and external valency k.

The case $i = 0$. Here $g_\gamma = 0$, i.e., γ is a tree. We choose the counterterm I^{CG}_{0k} to be zero because

- for trees the weight $w^{\gamma}_{\varepsilon L}$ has no singularity at $\varepsilon = 0$ and
- the sum over trees converges,

8.1. **A direct construction of counterterms.** One attempts to correct the fact that the limit

$$\lim_{\varepsilon \to 0} W_{\varepsilon L} II$$

does not exist, i.e., that the function of ε $W_{\varepsilon L} I$ is singular at $\varepsilon = 0$. A choice of the renormalization scheme Σ gives us a quantitative incarnation $Sing^{\Sigma}(W_{\varepsilon L} I)$ of this singularity. We will see that the main thing is that this measure of singularity $Sing^{\Sigma}$, is additive and idempotent.

The naive attempt to correct is by using as the counterterm $I^{\mathrm{C}}_{\varepsilon L}$ the object $Sing^{\Sigma}(W_{\varepsilon L}I)$ that causes the problem. One could expect that a problem in this strategy is a discrepancy between ε and L. However it actually works when we filter $\mathcal{I}_q$ by $\mathbb{N}^2$, the problem disappears because $Gr(W_{\varepsilon L})$ is identity.

Question. Does the strategy work when we make interval (ε, L) short, i.e., on the level of differential equations?

8.1.1. *Properties of the RGF operators $W_{\varepsilon L}$.*

$$W_P I \;=\; \hbar \log\left[e^{\hbar\partial_P} e^{\frac{1}{\hbar}I}\right].$$

8.1.2. *The graded version of the flow $W_{\varepsilon L}$ is identity.* For $\Theta\subseteq\mathbb{N}^2$ denote $I_\Theta \stackrel{\mathrm{def}}{=} \sum_{(i,k)\in\Theta} \hbar^i I_{ik}$ and $W_\Theta I \stackrel{\mathrm{def}}{=} \sum_{(i,k)\in\Theta} \hbar^i W_{ik} I \stackrel{\mathrm{def}}{=} \sum_{(i,k)\in\Theta} \hbar^i (WI)_{ik}$.

Lemma. Let $\alpha = (i,k) \in \mathbb{N}^2$.

(a) $W_{P,\alpha}(I) \;=\; W_{P,\alpha}(I_{\leq\alpha})$.

(b)

$$W_{P,\alpha}(I) \;=\; W_{P,\alpha}(I_{<\alpha}) \;+\; I_\alpha).$$

(c) $W_{\leq\alpha}$ is an automorphism of the $\mathcal{I}_{q_\alpha}$-torsor $\mathcal{I}_{q_{\leq\alpha}} \twoheadrightarrow \mathcal{I}_{q_{<\alpha}}$, i.e., for $I \in \mathcal{I}_{q_{\leq\alpha}}$ and $J \in \mathcal{I}_{q_\alpha}$

$$W_{P,\leq\alpha}(I + J_\alpha) \;=\; W_{P,\leq\alpha} I \;+\; J.$$

Sublemma. If $I_{r,s}$ appears in the tensor I_γ attached to some $\gamma \in cs\mathcal{FG}$ then $(g_\gamma, k_\gamma) \geq (r,s)$.

The equality happens for precisely one graph γ, the "(r,s)-*star*" graph $\star_{r,s}$ which consists of one internal vertex v. s edges from v to external vertices and the genus of v is r.

Proof of the sublemma. Clearly, $r \leq i$ since $r = g_v \leq g_\gamma = i$. So we only need to consider the case $r = i$ and check that $s \leq k$ and that $s = k$ iff $\gamma = \star_{r,s}$.

$r = i$ implies $g_v = g_\gamma$ and this tells us that

- $b_1(\gamma) = 0$, i.e., γ is a tree and
- inner vertices $u \neq v$ have genus 0, hence valency k_u is ≥ 3.

Now we will observe that the external valency of γ is

$$d_\gamma \;=\; k_v \;+\; \sum_{u\in \mathrm{iV}_\gamma - \{v\}} k_u - 2$$

where iV are the internal vertices. This implies the remaining claims of the sublemma. The formula is clear when presented as a picture but the long if we describe it in words.

For this we start with the subgraph $\widetilde{\gamma}$ of γ that consists of the vertex v and the edges from v. Since γ is a tree, subgraph $\widetilde{\gamma}$ has a "star" shape and γ is obtained from $\widetilde{\gamma}$ by attaching a tree T_e to the end of each edge e in $\widetilde{\gamma}$.

To get the summand k_v observe that the least that each tree T_e contributes is 1 and this happens iff T_e is a point, i.e., if it contains no internal vertices of γ. Then each internal vertex u that lies in T_e adds $k_u - 2$.

Proof of the lemma. We have

$$W_{P,\alpha}I = \int_{cs\mathcal{FG}_\alpha} w_P^\gamma I \quad \text{and} \quad w_P^\gamma I = \langle P_\gamma, I_\gamma \rangle$$

where tensor I_γ is obtained by putting at each internal vertex v the tensor I_{g_v,k_v}.

(a) follows from the sublemma.

(b) The difference $W_{P,\alpha}(I) - W_{P,\alpha}(I_{<\alpha})$ is given by the terms $w_P^\gamma I$ where $(g_\gamma, k_\kappa) = \alpha$ and I_γ features I_α. The sublemma says that there is just one such graph $\gamma = \star_\alpha$. For this graph, $w_P^\gamma I = I_\gamma$ since there are no internal edges, i.e., no contractions with the propagator P. Also, $I_\gamma = I_\alpha$ since γ has just one internal vertex and this is where I_α is positioned.

(c) follows from (a) and (b),

$$W_{P,\leq\alpha}(I+J) = W_{P,\leq\alpha}(I+J)_{<\alpha} + (I+J)_\alpha = W_{P,\leq\alpha}(I_{<\alpha}) + I_\alpha + J = W_{P,\leq\alpha}(I) + J.$$

8.1.3. *Inductive construction of counterterms.* By induction we assume that for $\beta < \alpha$ we have counterterms

$$I^{\mathrm{C}}(\varepsilon)_\beta \in \mathcal{O}_l(\mathcal{E}) \otimes \Sigma$$

such that for

$$I^{\mathrm{C}}(\varepsilon)_{<\alpha} = \sum_{(r,s)<\alpha} \hbar^r I^{\mathrm{C}}(\varepsilon)_{(r,s)}$$

and any $L > 0$, the function $WecL_{<\alpha}[I - I^{\mathrm{C}}(\varepsilon)_{<\alpha}]$ of $\varepsilon > 0$ is regular at $\varepsilon = 0$.[60]

Now we can define

$$I^{\mathrm{C}}(\varepsilon)_\alpha \stackrel{\text{def}}{=} Sing^\Sigma[\lim_{\varepsilon\to 0} W(\varepsilon, L)_{\leq\alpha}[I - I^{\mathrm{C}}(\varepsilon)_{<\alpha}] \in \mathcal{O}_l(\mathcal{E}) \otimes_{alg} \Sigma$$

and the $\leq \alpha$ counterterm

$$I^{\mathrm{C}}(\varepsilon)_{\leq\alpha} \stackrel{\text{def}}{=} I^{\mathrm{C}}(\varepsilon)_{<\alpha} + \hbar^i I^{\mathrm{C}}(\varepsilon)_\alpha.$$

According to the lemma 8.1.2.c

$$W(\varepsilon, L)_{\leq\alpha}[I - I^{\mathrm{C}}(\varepsilon)_{\leq\alpha}] = W(\varepsilon, L)_{\leq\alpha}[I - I^{\mathrm{C}}(\varepsilon)_{<\alpha}] - I^{\mathrm{C}}(\varepsilon)_\alpha.$$

[60] When $\alpha = (i, k)$ with $k = 0 < i$ then α has no direct predecessor so one needs to make a trivial "limit" observation.

This expression has a limit at $\varepsilon = 0$ since $Sing^\Sigma$ is additive and idempotent:

$$Sing^\Sigma[W(\varepsilon, L)_{\leq\alpha}[I - I^{\mathrm{C}}(\varepsilon)_{<\alpha}] \;=\; I^{\mathrm{C}}(\varepsilon)_\alpha = \; Sing^\Sigma[I^{\mathrm{C}}(\varepsilon)_\alpha].$$

Now $I^{\mathrm{C}}(\varepsilon) = \lim_{\alpha\to\infty} \; I^{\mathrm{C}}(\varepsilon)_{\leq\alpha}$ has the required properties.

8.1.4. *Independence of L.*

8.1.5. *Locality of I^{C}.* This follows from the independence of L.

8.1.6. *Uniqueness.* is also clear, i.e., the above inductive procedure is in each step the only possible one.

8.1.7. *Remark.* If $\lim_{\varepsilon\to 0} \; W_{\varepsilon L} I$ exists then the counterterm $I^{\mathrm{C}}(\varepsilon)$ vanishes.

Proof. For $\alpha = (i,k)$, the counterterm I^{C}_α is of the form $Sing^\Sigma[W(\varepsilon, L)_\alpha(I - I^{\mathrm{C}}_{<\alpha})]$. If we inductively know that $I^{\mathrm{C}}_{<\alpha} = 0$ then

$$Sing^\Sigma[W(\varepsilon, L)_\alpha(I - I^{\mathrm{C}}_{<\alpha})] = \; Sing^\Sigma(W(\varepsilon, L)_\alpha I) = \; 0$$

since the component $W(\varepsilon, L)_\alpha I$ of $W(\varepsilon, L)I$ has limit at $\varepsilon = 0$.

9. **Proof of the quantization theorem**

The construction $I \mapsto I[L]$ is easily seen to satisfy the conditions for an EQFT.

9.1. **From a scaled quantum interaction $I[-]$ to a single quantum interaction I.** We construct inductively I_α for $\alpha = (i,k)$. Assume that for $(r,s) < \alpha$ we have $I_{r,s}$ such that $I_{<\alpha} \stackrel{\mathrm{def}}{=} \sum_{(r,s)<\alpha} \; \hbar^i I_{r,s}$ reconstructs the the $<\alpha$-truncated $I[L]$ by the the $<\alpha$-truncated flow $W_{0,L}$:

$$W(0, L)_{<\alpha} \; I_{<\alpha} \;=\; I[L]_{<\alpha}.$$

The induction step is the following lemma:

Lemma. (a) The following quantity is local and independent of L

$$I_\alpha \;\stackrel{\mathrm{def}}{=}\; I_\alpha[L] \;-\; W(0, L)_\alpha \; I_{<\alpha}.$$

(b) $I_{\leq\alpha} \stackrel{\mathrm{def}}{=} I_{<\alpha} + \hbar^i I_\alpha$ satisfies

$$W(0, L)_{\leq\alpha} \; I_{\leq\alpha} \;=\; I[L]_{\leq\alpha}.$$

9.2. **The canonical description of $\mathcal{T}^{(\infty)}$ (i.e., without a choice of a renormalization scheme).**

Lemma. (a) The tangent space at $I[L]_0 \in \mathcal{T}^{(0)}$, is

$$T_{I[L]_0}\mathcal{T}^{(0)} \;\cong\; \mathcal{O}_l(\mathcal{E}).$$

10. **ZZZ**