Math 545 Linear transformations and the geometry of surfaces

A homework assignment

Let $S$ be a *smooth* surface in $\mathbb{R}^3$ given by the equation $f(x,y,z)=0$, where smoothness means that the gradient vector

$$\nabla f := \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

does not vanish at any point of $S$. Note that $\nabla f$ is a (non-linear in general) function from $\mathbb{R}^3$ to $\mathbb{R}^3$. The *tangent plane* $T_P S$ to $S$ at a point $P \in S$ is the two dimensional subspace of $\mathbb{R}^3$ orthogonal to the gradient vector $\nabla f(P)$. Note that we define the tangent plane $T_P S$ as a plane through the origin, which need not pass through $P$.

1. Let $\widetilde{S}$ be the unit sphere given by $x^2+y^2+z^2-1=0$ and $\widetilde{P}=(x_0,y_0,z_0)$ a point of $\widetilde{S}$. Show that the tangent plane $T_{\widetilde{P}}\widetilde{S}$ is the plane in $\mathbb{R}^3$ orthogonal to the vector $(x_0,y_0,z_0)$.

2. Let $S$ be the ellipsoid $(x/a)^2+(y/b)^2+(z/c)^2=3$, where $a,b,c$ are fixed positive numbers. Show that the point $P=(a,b,c)$ belongs to $S$ and the tangent plane of $S$ at $P$ is the plane cut out by the linear equation $x/a+y/b+z/c=0$.

3. A *parametrization* of an open subset of $S$ consists of an open subset $U$ of $\mathbb{R}^2$ together with a one-to-one map $X : U \to \mathbb{R}^3$, with the following properties.

   (a) The equality $f(X(u,v))=0$ holds, for all $(u,v)\in U$. This means that $X$ maps $U$ into $S$.

   (b) Write $X(u,v)=(x(u,v),y(u,v),z(u,v))$, expressing the components of $X$ as functions of the coordinates $u$ and $v$ on $U$. Then the entries of the the $3\times 2$ matrix

   $$dX(u,v) := \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

   are functions of $u$ and $v$, and we require $dX(u,v)$ to have rank 2, for every point $(u,v)$ in the open set $U$.

   Let $X_u$ be the first column of $dX(u,v)$ and $X_v$ the second column. Show that

   $$\beta := \{X_u(u_0,v_0),\ X_v(u_0,v_0)\} \tag{1}$$

   is a basis of $T_P S$ at the point $P := X(u_0,v_0)$, for every parametrization $X : U \to S$ and for every point $(u_0,v_0)$ of $U$.

4. Given two surfases $S$ and $\widetilde{S}$ and a "nice" map $G : S \to \widetilde{S}$, one can define a linear transformation $dG_P : T_P S \to T_{G(P)}\widetilde{S}$, called *the differential of* $G$ *at* $P$. We will not define it, but rather state how to compute $dG_P$ in terms of parametrizations of $S$ and $\widetilde{S}$. Note that the definition of $dG_P$ does not depend on the choice of

parametrizations (see, for example, section 2.4 of the book *Differential Geometry of Curves and Surfaces*, by M. P. DoCarmo, Prentice Hall 1976.)

Assume given a pair of parametrizations $X : U \to S$ and $\widetilde{X} : \widetilde{U} \to \widetilde{S}$, such that the image $G(X(U))$ is contained in the image $\widetilde{X}(\widetilde{U})$. Then given a point $(u_0, v_0)$ in $U$, there exists a unique point $(\tilde{u}_0, \tilde{v}_0)$ in $\widetilde{U}$, such that $\widetilde{X}(\tilde{u}_0, \tilde{v}_0) = G(X(u, v))$, since $\widetilde{X}$ is assumed to be one-to-one. Thus, there exists a unique function $g : U \to \widetilde{U}$, such that $\widetilde{X}(g(u, v)) = G(X(u, v))$, for all $(u, v) \in U$.

$$\begin{array}{ccc} S & \xrightarrow{G} & \widetilde{S} \\ X \uparrow & & \uparrow \widetilde{X} \\ U & \xrightarrow{g} & \widetilde{U}. \end{array}$$

Fix a point $(u_0, v_0)$ in $U$ and set $(\tilde{u}_0, \tilde{v}_0) := g(u_0, v_0)$. Set $P := X(u_0, v_0)$ and $\widetilde{P} := \widetilde{X}(\tilde{u}_0, \tilde{v}_0)$. Then $dG_P$ is defined (i.e., $G$ is "nice" at $P$) if the partials of $g$ are all defined at $(u_0, v_0)$. Express the components of $g$ as functions of $u$ and $v$ via the notation $g(u, v) = (\widetilde{u}(u, v), \widetilde{v}(u, v))$ and form the $2 \times 2$ matrix

$$dg \ := \ \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial v} \\ \\ \frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix}.$$

Then $\beta := \{X_u, X_v\}$, evaluated at $(u_0, v_0)$, is a basis of $T_P S$, $\tilde{\beta} := \{\widetilde{X}_{\tilde{u}}, \widetilde{X}_{\tilde{v}}\}$, evaluated at $(\tilde{u}_0, \tilde{v}_0)$, is a basis of $T_{\widetilde{P}} \widetilde{S}$, and $dg(u_0, v_0)$ is equal to the matrix $[[dG_P]]_{\beta, \tilde{\beta}}$ of the linear transformation $dG_P : T_P S \to T_{\widetilde{P}} \widetilde{S}$ with respect to these two bases. More explicitly,

$$\begin{aligned} dG_P(X_u) &= \frac{\partial \widetilde{X}}{\partial u} = \left(\frac{\partial \tilde{u}}{\partial u}\right) \widetilde{X}_{\tilde{u}} + \left(\frac{\partial \tilde{v}}{\partial u}\right) \widetilde{X}_{\tilde{v}} \ \text{ and} \\ dG_P(X_v) &= \frac{\partial \widetilde{X}}{\partial v} = \left(\frac{\partial \tilde{u}}{\partial v}\right) \widetilde{X}_{\tilde{u}} + \left(\frac{\partial \tilde{v}}{\partial v}\right) \widetilde{X}_{\tilde{v}}. \end{aligned}$$

The differential $dG_P$ can be defined independently of the choice of parametrizations, and the above equations say that *once parametrizations are chosen, $dG_P$ is compatible with the chain rule.*

5. Let $\widetilde{S}$ be the unit sphere in $\mathbb{R}^3$, given by the equation $x^2 + y^2 + z^2 = 1$. Let $S$ be a surface in $\mathbb{R}^3$, given by the equation $f(x, y, z) = 0$. The *Gauss map* $G : S \to \widetilde{S}$ of $S$ is given by

$$G(P) \ = \ \frac{1}{|\nabla f(P)|} \nabla f(P).$$

$G$ sends a point $P$ of $S$ to the point on the unit sphere corresponding to a unit normal vector to $S$ at $P$. Observe that the tangent plane $T_{G(P)} \widetilde{S}$ to the unit sphere is **equal** to $T_P S$, by part 1 above. Hence,

$$dG_P \ : \ T_P S \ \longrightarrow \ T_P S$$

is a linear transformation from $T_PS$ to **itself!** We can thus define the determinant $\det(dG_P)$ (Definition (18.7) on page 149 in our text). The determinant $\det(dG_P)$ is called the *Gaussian curvature of S at P*.

Let $S$ be the surface given by $x^2 + (y/2)^2 + (z/3)^2 - 3 = 0$. Let $U$ be the open subset of $\mathbb{R}^2$ given by $x^2 + (y/2)^2 < 3$. Let $X : U \to S$ be the parametrization of the upper half of the ellipsoid $S$, given by

$$X(u,v) = \left(u, v, 3\sqrt{3 - u^2 - (v/2)^2}\right).$$

Prove the equalities

$$X_u(u,v) = \begin{pmatrix} 1 \\ 0 \\ -9x/z \end{pmatrix} \quad \text{and} \quad X_v(u,v) = \begin{pmatrix} 0 \\ 1 \\ -9y/4z \end{pmatrix},$$

in $T_{(x,y,z)}S$, where $(x,y,z) = X(u,v)$.

6. Keep the notation of part 5. Choose the parametrization $\widetilde{X}(\tilde{u},\tilde{v}) = \left(\tilde{u}, \tilde{v}, \sqrt{1-\tilde{u}^2-\tilde{v}^2}\right)$ of $\widetilde{S}$, defined on the open unit disk $\widetilde{U}$ in $\mathbb{R}^2$. Set $P = (1,2,3)$. Then $\widetilde{P} = G(P) = \frac{1}{7}(6,3,2)$. Show that the matrix $[[dG_P]]_{\beta,\tilde{\beta}}$ of $dG_P$, with respect to the basis $\beta := \{X_u(1,2), X_v(1,2)\}$ of $T_PS$ and $\tilde{\beta} := \{\widetilde{X}_{\tilde{u}}(\frac{6}{7},\frac{3}{7}), \widetilde{X}_{\tilde{v}}(\frac{6}{7},\frac{3}{7})\}$ of $T_{\widetilde{P}}\widetilde{S}$, is equal to
$$\frac{3}{7^3}\begin{pmatrix} 34 & -5 \\ -32 & 22 \end{pmatrix}. \tag{2}$$

   Hint: Let $\tilde{\pi} : \widetilde{S} \to \widetilde{U}$ be the projection given by $\pi(x,y,z) = (x,y)$. Show first that the unique function $g : U \to \widetilde{U}$, satisfying $G(X(u,v)) = \widetilde{X}(g(u,v))$, is given in our case by $g(u,v) = \tilde{\pi}(G(X(u,v))) = \frac{3}{\sqrt{3+8u^2+(5/16)v^2}}\left(u, \frac{v}{4}\right)$.

7. Keep the notation of part 6. Show that the bases $\beta$ and $\tilde{\beta}$ of $T_PS$ are the same (this is a coincidence). Conclude that the matrix $[[dG_P]]_\beta$ of $dG_P$ with respect to the basis $\beta := \{X_u(1,2), X_v(1,2)\}$ of $T_PS$ is equal to the matrix in equation (2). Conclude also that the Gaussian curvature of $S$ at $P$ is $\frac{108}{7^4}$.

8. Let $W$ be the open subset $u^2 + (v/3)^2 < 3$ of $\mathbb{R}^2$ and $Y : W \to \mathbb{R}^3$ the function
$$Y(u,v) = (u, 2\sqrt{3 - u^2 - (v/3)^2}, v).$$

   Then $Y$ is another parametrization of an open subset of the ellipsoid $S$ in part 5 and $P = (1,2,3) = Y(1,3)$ is in the image of $Y$. Define the basis $\beta_2 := \{Y_u(1,3), Y_v(1,3)\}$ of $T_PS$ as in equation (1).

   Use your answer in part 7 and Theorem (13.6)' page 104 in the text in order to show that the matrix $[[dG_P]]_{\beta_2}$ of the differential $dG_P : T_PS \to T_PS$ of the Gauss map, with respect to the new basis $\beta_2$ of $T_PS$, is equal to
$$[[dG_P]]_{\beta_2} = \frac{3}{7^3}\begin{pmatrix} 44 & \frac{10}{3} \\ -18 & 12 \end{pmatrix}. \tag{3}$$

**The moral of this story:** The subspace $T_PS$ of $\mathbb{R}^3$, the **linear transformation** $dG_P : T_PS \to T_PS$, and the Gaussian curvature $\det(dG_P)$, do **not** depend on the choice of parametrization of $S$. In contrast, different parametrizations give rize to **different** $2 \times 2$ **matrices** of $dG_P$, such as (2), (3), or yet a third $2 \times 2$ matrix that would arise if we choose a parametrization of the ellipsoid $S$ via polar coordinates.