## EXERCISES

**1.** Verify that

(a) $(\sqrt{2}-i)-i(1-\sqrt{2}i)=-2i$; (b) $(2,-3)(-2,1)=(-1,8)$;

(c) $(3,1)(3,-1)\left(\frac{1}{5},\frac{1}{10}\right)=(2,1)$.

**2.** Show that

(a) $\operatorname{Re}(iz)=-\operatorname{Im} z$; (b) $\operatorname{Im}(iz)=\operatorname{Re} z$.

**3.** Show that $(1+z)^2=1+2z+z^2$.

**4.** Verify that each of the two numbers $z=1\pm i$ satisfies the equation $z^2-2z+2=0$.

**5.** Prove that multiplication of complex numbers is commutative, as stated at the beginning of Sec. 2.

**6.** Verify

(a) the associative law for addition of complex numbers, stated at the beginning of Sec. 2;

(b) the distributive law (3), Sec. 2.

**7.** Use the associative law for addition and the distributive law to show that

$$z(z_1+z_2+z_3)=zz_1+zz_2+zz_3.$$

**8.** (a) Write $(x,y)+(u,v)=(x,y)$ and point out how it follows that the complex number $0=(0,0)$ is unique as an additive identity.

(b) Likewise, write $(x,y)(u,v)=(x,y)$ and show that the number $1=(1,0)$ is a unique multiplicative identity.

**9.** Use $-1=(-1,0)$ and $z=(x,y)$ to show that $(-1)z=-z$.

**10.** Use $i=(0,1)$ and $y=(y,0)$ to verify that $-(iy)=(-i)y$. Thus show that the additive inverse of a complex number $z=x+iy$ can be written $-z=-x-iy$ without ambiguity.

**11.** Solve the equation $z^2+z+1=0$ for $z=(x,y)$ by writing

$$(x,y)(x,y)+(x,y)+(1,0)=(0,0)$$

and then solving a pair of simultaneous equations in $x$ and $y$.

*Suggestion:* Use the fact that no real number $x$ satisfies the given equation to show that $y\neq 0$.

*Ans.* $z=\left(-\frac{1}{2},\pm\frac{\sqrt{3}}{2}\right)$.

## 3. FURTHER PROPERTIES

In this section, we mention a number of other algebraic properties of addition and multiplication of complex numbers that follow from the ones already described

## EXERCISES

1. Reduce each of these quantities to a real number:

$$(a)\ \frac{1+2i}{3-4i}+\frac{2-i}{5i};\qquad (b)\ \frac{5i}{(1-i)(2-i)(3-i)};\qquad (c)\ (1-i)^4.$$

*Ans.* (a) $-2/5$; (b) $-1/2$; (c) $-4$.

2. Show that

$$\frac{1}{1/z}=z \qquad (z\neq 0).$$

3. Use the associative and commutative laws for multiplication to show that

$$(z_1z_2)(z_3z_4)=(z_1z_3)(z_2z_4).$$

4. Prove that if $z_1z_2z_3=0$, then at least one of the three factors is zero.

*Suggestion:* Write $(z_1z_2)z_3=0$ and use a similar result (Sec. 3) involving two factors.

5. Derive expression (6), Sec. 3, for the quotient $z_1/z_2$ by the method described just after it.

6. With the aid of relations (10) and (11) in Sec. 3, derive the identity

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right)=\frac{z_1z_2}{z_3z_4} \qquad (z_3\neq 0,\ z_4\neq 0).$$

7. Use the identity obtained in Exercise 6 to derive the cancellation law

$$\frac{z_1z}{z_2z}=\frac{z_1}{z_2} \qquad (z_2\neq 0,\ z\neq 0).$$

8. Use mathematical induction to verify the binomial formula (13) in Sec. 3. More precisely, note that the formula is true when $n=1$. Then, assuming that it is valid when $n=m$ where $m$ denotes any positive integer, show that it must hold when $n=m+1$.

*Suggestion:* When $n=m+1$, write

$$(z_1+z_2)^{m+1}=(z_1+z_2)(z_1+z_2)^m=(z_2+z_1)\sum_{k=0}^{m}\binom{m}{k}z_1^kz_2^{m-k}$$
$$=\sum_{k=0}^{m}\binom{m}{k}z_1^kz_2^{m+1-k}+\sum_{k=0}^{m}\binom{m}{k}z_1^{k+1}z_2^{m-k}$$

and replace $k$ by $k-1$ in the last sum here to obtain

$$(z_1+z_2)^{m+1}=z_2^{m+1}+\sum_{k=1}^{m}\left[\binom{m}{k}+\binom{m}{k-1}\right]z_1^kz_2^{m+1-k}+z_1^{m+1}.$$

**EXAMPLE 3.** If a point $z$ lies on the unit circle $|z| = 1$ about the origin, it follows from inequalities (7) and (8) that

$$|z - 2| \leq |z| + 2 = 3$$

and

$$|z - 2| \geq ||z| - 2| = 1.$$

The triangle inequality (4) can be generalized by means of mathematical induction to sums involving any finite number of terms:

$$(10) \qquad |z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n| \qquad (n = 2, 3, \ldots).$$

To give details of the induction proof here, we note that when $n = 2$, inequality (10) is just inequality (4). Furthermore, if inequality (10) is assumed to be valid when $n = m$, it must also hold when $n = m + 1$ since, by inequality (4),

$$\begin{aligned} |(z_1 + z_2 + \cdots + z_m) + z_{m+1}| &\leq |z_1 + z_2 + \cdots + z_m| + |z_{m+1}| \\ &\leq (|z_1| + |z_2| + \cdots + |z_m|) + |z_{m+1}|. \end{aligned}$$

## EXERCISES

**1.** Locate the numbers $z_1 + z_2$ and $z_1 - z_2$ vectorially when

*(a)* $z_1 = 2i, \quad z_2 = \dfrac{2}{3} - i$; *(b)* $z_1 = (-\sqrt{3}, 1), \quad z_2 = (\sqrt{3}, 0)$;

*(c)* $z_1 = (-3, 1), \quad z_2 = (1, 4)$; *(d)* $z_1 = x_1 + iy_1, \quad z_2 = x_1 - iy_1$.

**2.** Verify inequalities (3), Sec. 4, involving $\operatorname{Re} z$, $\operatorname{Im} z$, and $|z|$.

**3.** Use established properties of moduli to show that when $|z_3| \neq |z_4|$,

$$\frac{\operatorname{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

**4.** Verify that $\sqrt{2}\,|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$.

*Suggestion:* Reduce this inequality to $(|x| - |y|)^2 \geq 0$.

**5.** In each case, sketch the set of points determined by the given condition:

*(a)* $|z - 1 + i| = 1$; *(b)* $|z + i| \leq 3$; *(c)* $|z - 4i| \geq 4$.

**6.** Using the fact that $|z_1 - z_2|$ is the distance between two points $z_1$ and $z_2$, give a geometric argument that

*(a)* $|z - 4i| + |z + 4i| = 10$ represents an ellipse whose foci are $(0, \pm 4)$;

*(b)* $|z - 1| = |z + i|$ represents the line through the origin whose slope is $-1$.

An important identity relating the conjugate of a complex number $z = x + iy$ to its modulus is

$$z\overline{z} = |z|^2, \tag{7}$$

where each side is equal to $x^2 + y^2$. It suggests the method for determining a quotient $z_1/z_2$ that begins with expression (7), Sec. 3. That method is, of course, based on multiplying both the numerator and the denominator of $z_1/z_2$ by $\overline{z_2}$, so that the denominator becomes the real number $|z_2|^2$.

**EXAMPLE 1.** As an illustration,

$$\frac{-1+3i}{2-i} = \frac{(-1+3i)(2+i)}{(2-i)(2+i)} = \frac{-5+5i}{|2-i|^2} = \frac{-5+5i}{5} = -1+i.$$

See also the example in Sec. 3.

Identity (7) is especially useful in obtaining properties of moduli from properties of conjugates noted above. We mention that

$$|z_1 z_2| = |z_1||z_2| \tag{8}$$

and

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (z_2 \neq 0). \tag{9}$$

Property (8) can be established by writing

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 z_2)(\overline{z_1}\,\overline{z_2}) = (z_1\overline{z_1})(z_2\overline{z_2}) = |z_1|^2|z_2|^2 = (|z_1||z_2|)^2$$

and recalling that a modulus is never negative. Property (9) can be verified in a similar way.

**EXAMPLE 2.** Property (8) tells us that $|z^2| = |z|^2$ and $|z^3| = |z|^3$. Hence if $z$ is a point inside the circle centered at the origin with radius 2, so that $|z| < 2$, it follows from the generalized triangle inequality (10) in Sec. 4 that

$$|z^3 + 3z^2 - 2z + 1| \leq |z|^3 + 3|z|^2 + 2|z| + 1 < 25.$$

## EXERCISES

**1.** Use properties of conjugates and moduli established in Sec. 5 to show that

*(a)* $\overline{\overline{z} + 3i} = z - 3i$; *(b)* $\overline{iz} = -i\overline{z}$;

*(c)* $\overline{(2+i)^2} = 3 - 4i$; *(d)* $|(2\overline{z} + 5)(\sqrt{2} - i)| = \sqrt{3}\,|2z + 5|$.

**2.** Sketch the set of points determined by the condition

*(a)* $\text{Re}(\overline{z} - i) = 2$; *(b)* $|2\overline{z} + i| = 4$.

**3.** Verify properties (3) and (4) of conjugates in Sec. 5.

**4.** Use property (4) of conjugates in Sec. 5 to show that

*(a)* $\overline{z_1 z_2 z_3} = \overline{z_1}\,\overline{z_2}\,\overline{z_3}$; *(b)* $\overline{z^4} = \overline{z}^4$.

**5.** Verify property (9) of moduli in Sec. 5.

**6.** Use results in Sec. 5 to show that when $z_2$ and $z_3$ are nonzero,

*(a)* $\overline{\left(\dfrac{z_1}{z_2 z_3}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}\,\overline{z_3}}$; *(b)* $\left|\dfrac{z_1}{z_2 z_3}\right| = \dfrac{|z_1|}{|z_2||z_3|}$.

**7.** Show that

$$|\text{Re}(2 + \overline{z} + z^3)| \le 4 \qquad \text{when } |z| \le 1.$$

**8.** It is shown in Sec. 3 that if $z_1 z_2 = 0$, then at least one of the numbers $z_1$ and $z_2$ must be zero. Give an alternative proof based on the corresponding result for real numbers and using identity (8), Sec. 5.

**9.** By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using inequality (8), Sec. 4, show that if $z$ lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \le \frac{1}{3}.$$

**10.** Prove that

*(a)* $z$ is real if and only if $\overline{z} = z$;

*(b)* $z$ is either real or pure imaginary if and only if $\overline{z}^2 = z^2$.

**11.** Use mathematical induction to show that when $n = 2, 3, \ldots$,

*(a)* $\overline{z_1 + z_2 + \cdots + z_n} = \overline{z_1} + \overline{z_2} + \cdots + \overline{z_n}$; *(b)* $\overline{z_1 z_2 \cdots z_n} = \overline{z_1}\,\overline{z_2} \cdots \overline{z_n}$.

**12.** Let $a_0, a_1, a_2, \ldots, a_n$ $(n \ge 1)$ denote *real* numbers, and let $z$ be any complex number. With the aid of the results in Exercise 11, show that

$$\overline{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n} = a_0 + a_1\overline{z} + a_2\overline{z}^2 + \cdots + a_n\overline{z}^n.$$

**13.** Show that the equation $|z - z_0| = R$ of a circle, centered at $z_0$ with radius $R$, can be written

$$|z|^2 - 2\,\text{Re}(z\overline{z_0}) + |z_0|^2 = R^2.$$

**14.** Using expressions (6), Sec. 5, for Re $z$ and Im $z$, show that the hyperbola $x^2 - y^2 = 1$ can be written

$$z^2 + \overline{z}^2 = 2.$$

**15.** Follow the steps below to give an algebraic derivation of the triangle inequality (Sec. 4)

$$|z_1 + z_2| \le |z_1| + |z_2|.$$

*(a)* Show that

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2}.$$

and, since (Sec. 7)

$$z_2^{-1} = \frac{1}{r_2} e^{-i\theta_2},$$

one can see that

$$\arg(z_2^{-1}) = -\arg z_2. \tag{3}$$

Hence

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \tag{4}$$

Statement (3) is, of course, to be interpreted as saying that the set of all values on the left-hand side is the same as the set of all values on the right-hand side. Statement (4) is, then, to be interpreted in the same way that statement (2) is.

**EXAMPLE 2.** In order to find the principal argument Arg $z$ when

$$z = \frac{-2}{1+\sqrt{3}i},$$

observe that

$$\arg z = \arg(-2) - \arg(1+\sqrt{3}i).$$

Since

$$\operatorname{Arg}(-2) = \pi \quad \text{and} \quad \operatorname{Arg}(1+\sqrt{3}i) = \frac{\pi}{3},$$

one value of $\arg z$ is $2\pi/3$; and, because $2\pi/3$ is between $-\pi$ and $\pi$, we find that $\operatorname{Arg} z = 2\pi/3$.

## EXERCISES

1. Find the principal argument Arg $z$ when

   *(a)* $z = \dfrac{i}{-2-2i}$; *(b)* $z = (\sqrt{3}-i)^6$.

   *Ans.* *(a)* $-3\pi/4$; *(b)* $\pi$.

2. Show that *(a)* $|e^{i\theta}| = 1$; *(b)* $\overline{e^{i\theta}} = e^{-i\theta}$.

3. Use mathematical induction to show that

   $$e^{i\theta_1} e^{i\theta_2} \cdots e^{i\theta_n} = e^{i(\theta_1+\theta_2+\cdots+\theta_n)} \qquad (n = 2, 3, \ldots).$$

4. Using the fact that the modulus $|e^{i\theta} - 1|$ is the distance between the points $e^{i\theta}$ and 1 (see Sec. 4), give a geometric argument to find a value of $\theta$ in the interval $0 \le \theta < 2\pi$ that satisfies the equation $|e^{i\theta} - 1| = 2$.

   *Ans.* $\pi$.

**5.** By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

(*a*) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$; (*b*) $5i/(2+i) = 1+2i$;

(*c*) $(-1+i)^7 = -8(1+i)$; (*d*) $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$.

**6.** Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2,$$

where principal arguments are used.

**7.** Let $z$ be a nonzero complex number and $n$ a negative integer $(n = -1, -2, \ldots)$. Also, write $z = re^{i\theta}$ and $m = -n = 1, 2, \ldots$. Using the expressions

$$z^m = r^m e^{im\theta} \quad \text{and} \quad z^{-1} = \left(\frac{1}{r}\right) e^{i(-\theta)},$$

verify that $(z^m)^{-1} = (z^{-1})^m$ and hence that the definition $z^n = (z^{-1})^m$ in Sec. 7 could have been written alternatively as $z^n = (z^m)^{-1}$.

**8.** Prove that two nonzero complex numbers $z_1$ and $z_2$ have the same moduli if and only if there are complex numbers $c_1$ and $c_2$ such that $z_1 = c_1 c_2$ and $z_2 = c_1\overline{c_2}$.

*Suggestion:* Note that

$$\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\exp\left(i\frac{\theta_1-\theta_2}{2}\right) = \exp(i\theta_1)$$

and [see Exercise 2(*b*)]

$$\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\overline{\exp\left(i\frac{\theta_1-\theta_2}{2}\right)} = \exp(i\theta_2).$$

**9.** Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1)$$

and then use it to derive *Lagrange's trigonometric identity:*

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(2n+1)\theta/2]}{2\sin(\theta/2)} \qquad (0 < \theta < 2\pi).$$

*Suggestion:* As for the first identity, write $S = 1 + z + z^2 + \cdots + z^n$ and consider the difference $S - zS$. To derive the second identity, write $z = e^{i\theta}$ in the first one.

**10.** Use de Moivre's formula (Sec. 7) to derive the following trigonometric identities:

(*a*) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$; (*b*) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

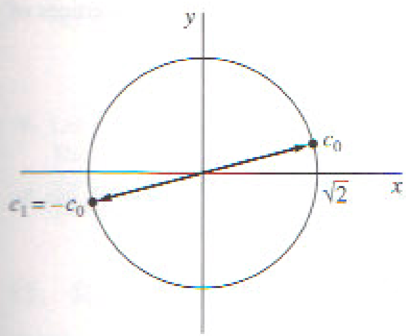


FIGURE 14

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}, \quad \sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} \tag{4}$$

enable us to write

$$\cos^2\frac{\pi}{12} = \frac{1}{2}\left(1+\cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right) = \frac{2+\sqrt{3}}{4},$$

$$\sin^2\frac{\pi}{12} = \frac{1}{2}\left(1-\cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right) = \frac{2-\sqrt{3}}{4}.$$

Consequently,

$$c_0 = \sqrt{2}\left(\sqrt{\frac{2+\sqrt{3}}{4}} + i\sqrt{\frac{2-\sqrt{3}}{4}}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right).$$

Since $c_1 = -c_0$, the two square roots of $\sqrt{3}+i$ are, then,

$$\pm\frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right). \tag{5}$$

## EXERCISES

1. Find the square roots of *(a)* $2i$; *(b)* $1-\sqrt{3}i$ and express them in rectangular coordinates.

   *Ans.* *(a)* $\pm(1+i)$; *(b)* $\pm\dfrac{\sqrt{3}-i}{\sqrt{2}}$.

2. In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain squares, and point out which is the principal root:

   *(a)* $(-16)^{1/4}$; *(b)* $(-8-8\sqrt{3}i)^{1/4}$.

   *Ans.* *(a)* $\pm\sqrt{2}(1+i)$, $\pm\sqrt{2}(1-i)$; *(b)* $\pm(\sqrt{3}-i)$, $\pm(1+\sqrt{3}i)$.

**3.** In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

*(a)* $(-1)^{1/3}$; *(b)* $8^{1/6}$.

*Ans. (b)* $\pm\sqrt{2},\ \pm\dfrac{1+\sqrt{3}i}{\sqrt{2}},\ \pm\dfrac{1-\sqrt{3}i}{\sqrt{2}}$.

**4.** According to Sec. 9, the three cube roots of a nonzero complex number $z_0$ can be written $c_0, c_0\omega_3, c_0\omega_3^2$ where $c_0$ is the principal cube root of $z_0$ and

$$\omega_3 = \exp\left(i\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}i}{2}.$$

Show that if $z_0 = -4\sqrt{2} + 4\sqrt{2}i$, then $c_0 = \sqrt{2}(1+i)$ and the other two cube roots are, in rectangular form, the numbers

$$c_0\omega_3 = \frac{-(\sqrt{3}+1) + (\sqrt{3}-1)i}{\sqrt{2}}, \qquad c_0\omega_3^2 = \frac{(\sqrt{3}-1) - (\sqrt{3}+1)i}{\sqrt{2}}.$$

**5.** *(a)* Let $a$ denote any fixed real number and show that the two square roots of $a+i$ are

$$\pm\sqrt{A}\exp\left(i\frac{\alpha}{2}\right)$$

where $A = \sqrt{a^2+1}$ and $\alpha = \operatorname{Arg}(a+i)$.

*(b)* With the aid of the trigonometric identities (4) in Example 3 of Sec. 10, show that the square roots obtained in part *(a)* can be written

$$\pm\frac{1}{\sqrt{2}}\left(\sqrt{A+a} + i\sqrt{A-a}\right).$$

(Note that this becomes the final result in Example 3, Sec. 10, when $a = \sqrt{3}$.)

**6.** Find the four zeros of the polynomial $z^4 + 4$, one of them being

$$z_0 = \sqrt{2}\,e^{i\pi/4} = 1 + i.$$

Then use those zeros to factor $z^2 + 4$ into quadratic factors with real coefficients.

*Ans.* $(z^2 + 2z + 2)(z^2 - 2z + 2)$.

**7.** Show that if $c$ is any $n$th root of unity other than unity itself, then

$$1 + c + c^2 + \cdots + c^{n-1} = 0.$$

*Suggestion:* Use the first identity in Exercise 9, Sec. 8.

**8.** *(a)* Prove that the usual formula solves the quadratic equation

$$az^2 + bz + c = 0 \qquad (a \neq 0)$$

when the coefficients $a$, $b$, and $c$ are complex numbers. Specifically, by completing the square on the left-hand side, derive the *quadratic formula*

$$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a},$$

where both square roots are to be considered when $b^2 - 4ac \neq 0$, *(b)* Use the result in part *(a)* to find the roots of the equation $z^2 + 2z + (1 - i) = 0$.

*Ans. (b)* $\left(-1 + \frac{1}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}}, \quad \left(-1 - \frac{1}{\sqrt{2}}\right) - \frac{i}{\sqrt{2}}$.

**9.** Let $z = re^{i\theta}$ be a nonzero complex number and $n$ a negative integer ($n = -1, -2, \ldots$). Then define $z^{1/n}$ by means of the equation $z^{1/n} = (z^{-1})^{1/m}$ where $m = -n$. By showing that the $m$ values of $(z^{1/m})^{-1}$ and $(z^{-1})^{1/m}$ are the same, verify that $z^{1/n} = (z^{1/m})^{-1}$. (Compare with Exercise 7, Sec. 8.)

## 11. REGIONS IN THE COMPLEX PLANE

In this section, we are concerned with sets of complex numbers, or points in the $z$ plane, and their closeness to one another. Our basic tool is the concept of an $\varepsilon$ *neighborhood*

(1) $$|z - z_0| < \varepsilon$$

of a given point $z_0$. It consists of all points $z$ lying inside but not on a circle centered at $z_0$ and with a specified positive radius $\varepsilon$ (Fig. 15). When the value of $\varepsilon$ is understood or is immaterial in the discussion, the set (1) is often referred to as just a neighborhood. Occasionally, it is convenient to speak of a *deleted neighborhood*, or punctured disk,

(2) $$0 < |z - z_0| < \varepsilon$$

consisting of all points $z$ in an $\varepsilon$ neighborhood of $z_0$ except for the point $z_0$ itself.

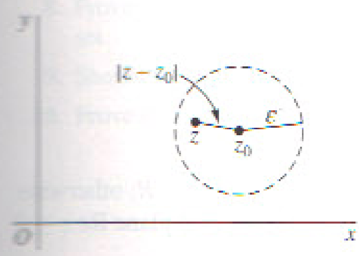


**FIGURE 15**

A point $z_0$ is said to be an *interior point* of a set $S$ whenever there is some neighborhood of $z_0$ that contains only points of $S$; it is called an *exterior point* of $S$ when there exists a neighborhood of it containing no points of $S$. If $z_0$ is neither of these, it is a *boundary point* of $S$. A boundary point is, therefore, a point all of whose neighborhoods contain at least one point in $S$ and at least one point not in $S$. The totality of all boundary points is called the *boundary* of $S$. The circle $|z| = 1$, for instance, is the boundary of each of the sets

(3) $$|z| < 1 \quad \text{and} \quad |z| \leq 1.$$

A set is *open* if it contains none of its boundary points. It is left as an exercise to show that a set is open if and only if each of its points is an interior point. A set is *closed* if it contains all of its boundary points, and the *closure* of a set $S$ is the closed set consisting of all points in $S$ together with the boundary of $S$. Note that the first of the sets (3) is open and that the second is its closure.

Some sets are, of course, neither open nor closed. For a set to be not open, there must be a boundary point that is contained in the set; and if a set is not closed, there exists a boundary point not contained in the set. Observe that the punctured disk $0 < |z| \leq 1$ is neither open nor closed. The set of all complex numbers is, on the other hand, both open and closed since it has no boundary points.

An open set $S$ is *connected* if each pair of points $z_1$ and $z_2$ in it can be joined by a *polygonal line*, consisting of a finite number of line segments joined end to end, that lies entirely in $S$. The open set $|z| < 1$ is connected. The annulus $1 < |z| < 2$ is, of course, open and it is also connected (see Fig. 16). A nonempty open set that is connected is called a *domain*. Note that any neighborhood is a domain. A domain together with some, none, or all of its boundary points is referred to as a *region*.

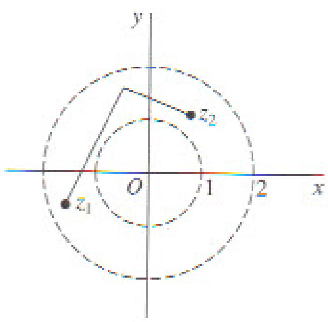


**FIGURE 16**

A set $S$ is *bounded* if every point of $S$ lies inside some circle $|z| = R$; otherwise, it is *unbounded*. Both of the sets (3) are bounded regions, and the half plane $\text{Re}\, z \geq 0$ is unbounded.

A point $z_0$ is said to be an *accumulation point* of a set $S$ if each deleted neighborhood of $z_0$ contains at least one point of $S$. It follows that if a set $S$ is closed, then it contains each of its accumulation points. For if an accumulation point $z_0$ were not in $S$, it would be a boundary point of $S$; but this contradicts the fact that a closed set contains all of its boundary points. It is left as an exercise to show that the converse is, in fact, true. Thus a set is closed if and only if it contains all of its accumulation points.

Evidently, a point $z_0$ is *not* an accumulation point of a set $S$ whenever there exists some deleted neighborhood of $z_0$ that does not contain at least one point of $S$. Note that the origin is the only accumulation point of the set $z_n = i/n$ $(n = 1, 2, \ldots)$.



**EXERCISES**

1. Sketch the fol
(a) $|z - 2 + i$
(c) $\text{Im}\, z > 1$;
(e) $0 \leq \arg z$
*Ans. (b),*
2. Which sets in
*Ans. (e).*
3. Which sets in
*Ans. (a).*
4. In each case,
(a) $-\pi < \arg$
(c) $\text{Re}\left(\frac{1}{z}\right) \leq$
5. Let $S$ be the c
why $S$ is not
6. Show that a se
7. Determine the
(a) $z_n = i^n$ (n
(c) $0 \leq \arg z$
*Ans. (a)*
8. Prove that if a
set.
9. Show that any
10. Prove that a fi