

Name: My SolutionSolve 5 of the following 6 problems. Please do **not** grade problem ____.

1. (20 points) Compute the integral $\int_C f(z)dz$, where $f(x+iy) = x^2 + iy$ and C is the straight line segment from 0 to $1+i$.

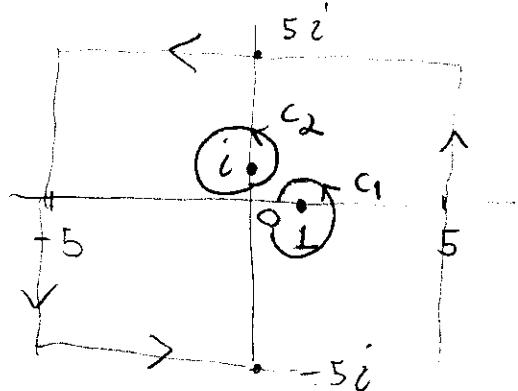
Parametrization of C : $z(t) = t + it$, $0 \leq t \leq 1$.

$$\begin{aligned} \int_C f(z)dz &= \int_0^1 f(z(t)) \cdot \frac{dz}{dt} dt = \int_0^1 (t^2 + it)(1+i) dt \\ &= \left[\left(\frac{t^3}{3} - \frac{t^2}{2} \right) + i \left(\frac{t^2}{2} + \frac{t^3}{3} \right) \right]_0^1 = \left(\frac{1}{3} - \frac{1}{2} \right) + i \left(\frac{1}{2} + \frac{1}{3} \right) = \\ &= \boxed{-\frac{1}{6} + \frac{5}{6}i} \end{aligned}$$

Also correct: $(1+i)\left(\frac{1}{3} + \frac{i}{2}\right)$

2. (20 points) Let C be the square with vertices at the points $\pm 5 \pm 5i$ (oriented counterclockwise). Compute $\int_C \frac{z^4 dz}{(z-1)^2(z-i)}$

$$\text{Let } f(z) = \frac{z^4}{(z-1)^2(z-i)}$$



The function f is analytic at all points of \mathbb{C} other than $z=1, z=i$.

Let C_1 be a circle of radius $\frac{1}{2}$ centered at $z=1$.
Let C_2 " " " " " " $\frac{1}{2}$ " at $z=i$.

The function f is analytic at all points z , on or interior to C_1 , but not interior to any C_i . Cauchy-Goursat Theorem for MULTI-CONNECTED domains yields,

$$I := \int_C f(z) dz = \underbrace{\int_{C_1} f(z) dz}_{I_1} + \underbrace{\int_{C_2} f(z) dz}_{I_2}.$$

$$I_1 = \int_{C_1} \frac{z^4/(z-i)}{(z-1)^2} dz = 2\pi i \cdot \frac{\partial}{\partial z} \left(\frac{z^4}{z-i} \right) \Big|_{z=1} = 2\pi i \left(\frac{4z^3(z-i)-z^4}{(z-i)^2} \right) \Big|_{z=1} = 2\pi i \frac{3-4i}{(1-i)^2}$$

Cauchy's Integral Formula

$$I_2 = \int_{C_2} \frac{z^4/(z-1)^2}{z-i} dz = 2\pi i \frac{(i)^4}{(i-1)^2} = 2\pi i \frac{1}{(1-i)^2}$$

$$I = I_1 + I_2 = (2\pi i) \frac{1-4i}{(1-i)^2} = 8\pi i \frac{(1+i)}{2} = 4\pi(-1+i) = (-4+4i)$$

3. (20 points) Let C be the circle of radius 2 centered at the origin (traversed counter-clockwise). Compute the integral $\int_C \frac{e^{3z}}{(z-1)^k} dz$, for all integers k (positive, zero, or negative). Justify your answer!

If $k \leq 0$, then $f(z) = \frac{e^{3z}}{(z-1)^k}$ is analytic at all points of G and all points interior to C .

Hence $\int_C f(z) dz = 0$ by Cauchy-Goursat theorem.

$$\text{For } k \geq 1, \int_C \frac{e^{3z}}{(z-1)^k} dz = g^{(k-1)}(1) = 3^{(k-1)} e^3$$

$$g^{(m)}(z) = 3^m e^{3z}$$

$$g^{(m)}(1) = 3^m e^3$$

$$\text{So } \int_C \frac{e^{3z}}{(z-1)^k} dz = \frac{(2+i) \cdot 3}{(k-1)!} e^3$$

4. (20 points) Determine whether the following statements are true or false. **Justify your answers!**

a) Let C be a closed contour, which does not pass through the origin. Then $\int_C \frac{dz}{z^2} = 0$.

True.

$F(z) = -\frac{1}{z}$ is an anti-derivative of $f(z) = \frac{1}{z^2}$, over $\mathbb{C} \setminus \{0\}$. Hence, $\int_C \frac{1}{z^2} dz = 0$, for every closed contour in $\mathbb{C} \setminus \{0\}$.

- 7 pts b) If $f(z)$ is an entire function, C is the unit circle traversed counterclockwise, and $|z_0| < 1$, then $\int_C \frac{f'(z)}{z - z_0} dz = \int_C \frac{f(z)}{(z - z_0)^2} dz$. True

Cauchy's (()) Integral
Formula for the first
derivative of the
function $f(z)$
at z_0 .

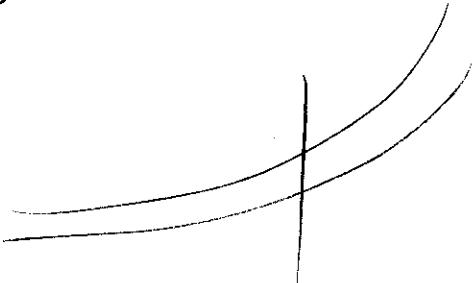
Cauchy's Integral
Formula for the first
derivative of the
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at z_0 .

c) Let α and β be arbitrary complex numbers. For any path C from α to β we have $\int_C \bar{z} dz = (\bar{\beta}^2 - \bar{\alpha}^2)/2$.

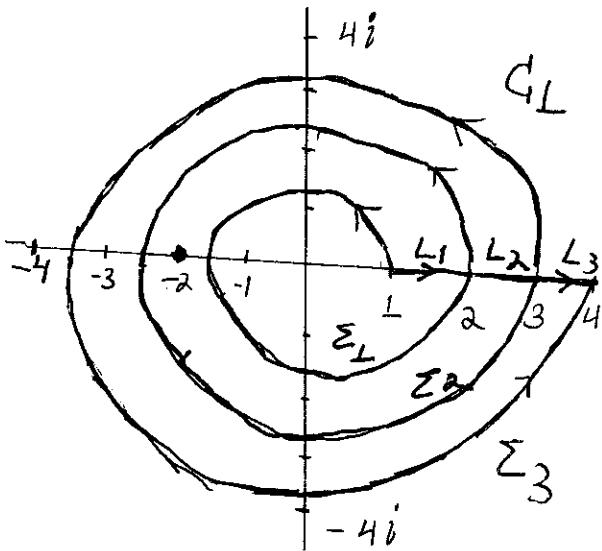
(6) Pto False, ($f(z) = \bar{z}$ is not analytic),
 Counter example:
 Take $\alpha = 0$, $\beta = i$, the straight line given by
 $z(t) = ti$, $0 \leq t \leq 1$,

$$\int_C \bar{z} dz = \int_0^1 \bar{z}(t) \frac{dz}{dt} dt = \int_0^1 t dt = \left[\frac{t^2}{2} \right]_0^1 = \frac{1}{2},$$

$$\frac{1}{2} (\bar{\beta}^2 - \bar{\alpha}^2) = -\frac{1}{2}$$



5. (20 points) Let C_1 be the contour given by the parametrization $z(t) = (1+t)e^{2\pi it}$, $0 \leq t \leq 3$. Let C_2 be the line segment from 1 to 4 parametrized by $z(t) = 1+t$, $0 \leq t \leq 3$. Compute the difference $\int_{C_1} \frac{dz}{z+2} - \int_{C_2} \frac{dz}{z+2}$. Justify your answer!



$C_L = \Sigma_1 + \Sigma_2 + \Sigma_3$, where
 Σ_2 is the segment of L starting at 2 and ending at $3 \in \mathbb{R}$.

$$C_2 = L_1 + L_2 + L_3, \text{ where}$$

L_2 is the line segment $\{x+0i, 2 \leq x \leq 3\}$

$P_2 = C_2 - L_2$ is a simple closed contour

II



The point $z = -2$ is interior to P_2 and P_3 but NOT to P_1 . Hence, $f(z) = \frac{1}{z+2}$ is analytic on P_1 and at any point of its interior. Thus

$\int_{P_1} f(z) dz = 0$. On the other hand, Cauchy's Integral

Formula yields, $\int_{P_1 + P_2 + P_3} \frac{1}{z+2} dz = 2\pi i \cdot (\text{value of constant function}) =$

Hence, $\int_{C_1 + C_2} \frac{dz}{z+2} = \int_{P_1 + P_2 + P_3} \frac{dz}{z+2} = 0 + 2\pi i + 2\pi i = \boxed{4\pi i}$

$= 2\pi i$
for $r = 2, 3$.

6. (20 points) Let u be a harmonic function defined (and having partials of all order) on the whole of \mathbb{R}^2 . Assume that the first partials of u satisfy the inequality

$$(u_x)^2 + (u_y)^2 \leq 7.$$

Prove that u must be a linear function, i.e., $u(x, y) = ax + by + c$, for some constant real numbers a, b , and c .

Hint: Consider the function $f(x + iy) = u_x(x, y) - iu_y(x, y)$.

Step 1: (The function f is analytic)

$f(x+iy) = \tilde{u}(x,y) + i\tilde{v}(x,y)$, where $\tilde{u} = u_x$ and $\tilde{v} = -u_y$. \tilde{u} and \tilde{v} have partials of all order, since u, v have by assumption.

$$\tilde{u}_x = u_{xx} = -u_{yy} = \tilde{v}_y$$

Since u is harmonic!!

$$\tilde{u}_y = u_{xy} = u_{yx} = -\tilde{v}_x.$$

Partials commute

Hence the function f satisfies the Cauchy-Riemann Equations. We conclude that f is analytic on the whole complex plane (entire).

Step 2: The function f is a bounded entire function, hence constant. Thus, $f(x+iy) = a - ib$ for some $a, b \in \mathbb{R}$, for all x, y .

Step 3: $u_x = a, u_y = b$, are constant, by step 2.

Thus, $h(x,y) := u(x,y) - ax - by$ has vanishing first partial.

It follows that $h(x,y) \equiv c$ is constant. Thus

$$u(x,y) = \cancel{\quad} \quad ax + by + c.$$