# Math 421 Midterm 2 Spring 2014

Solve 5 of the following 6 problems. Please do **not** grade problem __.

1. (20 points) Compute the integral $\int_C f(z)dz$, where $f(x+iy) = x^2 + iy$ and $C$ is the straight line segment from 0 to $1+i$.

2. (20 points) Let $C$ be the square with vertices at the points $\pm 5 \pm 5i$ (oriented counterclockwise). Compute $\int_C \frac{z^4 dz}{(z-1)^2(z-i)}$

3. (20 points) Let $C$ be the circle of radius 2 centered at the origin (traversed counterclockwise). Compute the integral $\int_C \frac{e^{3z}}{(z-1)^k}dz$, for all integers $k$ (positive, zero, or negative). **Justify your answer!**

4. (20 points) Determine whether the following statements are true or false. **Justify your answers!**
   **a)** Let $C$ be a closed contour, which does not pass through the origin. Then $\int_C \frac{dz}{z^2} = 0$.
   **b)** If $f(z)$ is an entire function, $C$ is the unit circle traversed counterclockwise, and $|z_0| < 1$, then $\int_C \frac{f'(z)}{z-z_0}dz = \int_C \frac{f(z)}{(z-z_0)^2}dz$.
   **c)** Let $\alpha$ and $\beta$ be arbitrary complex numbers. For any path $C$ from $\alpha$ to $\beta$ we have $\int_C \bar{z}dz = (\bar{\beta}^2 - \bar{\alpha}^2)/2$.

5. (20 points) Let $C_1$ be the contour given by the parametrization $z(t) = (1+t)e^{2\pi i t}$, $0 \le t \le 3$. Let $C_2$ be the line segment from 1 to 4 parametrized by $z(t) = 1+t$, $0 \le t \le 3$. Compute the difference $\int_{C_1} \frac{dz}{z+2} - \int_{C_2} \frac{dz}{z+2}$. **Justify your answer!**

6. (20 points) Let $u$ be a harmonic function defined (and having partials of all order) on the whole of $\mathbb{R}^2$. Assume that the first partials of $u$ satisfy the inequality
$$(u_x)^2 + (u_y)^2 \le 7.$$
Prove that $u$ must be a linear function, i.e., $u(x,y) = ax + by + c$, for some constant real numbers $a$, $b$, and $c$.
Hint: Consider the function $f(x+iy) = u_x(x,y) - iu_y(x,y)$.