Math 421 Midterm 1 Spring 2014

Name:__________________________

1. (35 points) Compute the following (in cartesian or polar form):

   a) The polar form of $z := \dfrac{3}{1+i}$.

   b) $|z^5|$, where z is given in part a.

   c) $\mathrm{Log}(z^9)$, where z is given in part a.

   d) Find all values of $(-1+\sqrt{3}i)^{(1/5)}$. How many different values are there?

   e) Find all values of $(1-i)^{(1+i)}$. How many different values are there?

2. (15 points) Prove that the following function is analytic on the whole complex plane $f(x+iy) = (e^{x+y}+e^{-x-y})\cos(y-x) + i(e^{x+y}-e^{-x-y})\sin(y-x)$. Carefully state any theorem you use.

3. (a) (5 points) Let λ be a complex number. Show that the equation $w + \dfrac{1}{w} = 2\lambda$ in the unknown w has at least one non-zero complex solution and at most two such solutions. Express the solutions in terms of λ. Hint: reduce to a quadratic equation.

   (b) (15 points) Show that the function $\cos : \mathbb{C} \to \mathbb{C}$ is onto, i.e., that every complex number λ is a value of cos. Hint: Use part 3a.

4. (20 points) a) Prove that the function $u(x,y) = x^3 - 3xy^2 + 6x^2y - 2y^3 + \ln(x^2+y^2)$ is harmonic on $\mathbb{R}^2 \setminus \{(0,0)\}$ (on the plane minus the origin).

   b) Find a harmonic conjugate v of the function $u(x,y)$ in the upper half plane $\mathbb{H} := \{x+iy \text{ such that } y > 0\}$. Hint: $x^2+y^2 = |z|^2$.

   c) Find an analytic function $f(z)$ on the upper half plane $\mathbb{H}$, such that $Re(f)$ is the Harmonic function u in part a. Your answer must be expressed as a function of $z = x+iy$, not x and y.

5. (10 points) Let U be the open subset of the complex plane above the line $y = x$, above the line $y = -x$, and inside the unit circle

   $$U \;=\; \{x+iy \text{ such that } y > x \text{ and } y > -x \text{ and } x^2+y^2 < 1\}.$$

   Describe geometrically the image $\mathrm{Log}(U)$ of U via the principal branch of the logarithm function. Draw the image and describe it in words. **Justify your answer!**