Math $421 \quad$ Final Exam Fall 2002
Name: $\qquad$
Solve problem 1 and 7 out of problems 2 to 9 . If you solve all 9, then problem 9 will not be graded. Please fill in: Please do not grade Problem number $\qquad$ .

1. (16 points) a) Show that the Laurent series of $\frac{1}{\sin (z)}$, centered at 0 , has the form

$$
\begin{equation*}
\frac{1}{\sin (z)}=\frac{1}{z}+\frac{1}{6} z+\frac{7}{360} z^{3}+\cdots \text { terms of order at least five. } \tag{1}
\end{equation*}
$$

(You can use equality (1) in the subsequent parts, even if you do not derive it).
b) Find the principal part at $z=0$ of the function $f(z)=\frac{1-z}{z^{5} \cdot \sin (z)}$
c) Find all the singularities of $f(z)$ (given in part b) in the disk $\{|z|<4\}$ and determine their type (isolated, removable, pole of what order, essential).
d) Find the residue at each isolated singularity in $D$.
2. (12 points) a) Compute $\sin \left(\frac{\pi}{4}+i \ln (3)\right)$. Simplify your answer as much as possible. b) Find all solutions of the equation $\cos (z)=i$.
3. (12 points) Compute the integral $\int_{C} \frac{\sin (z)+1}{e^{3 z}-e^{z}} d z$, where $C$ is the circle $\{|z|=1\}$ traversed counterclockwise.
4. (12 points) a) Find the Taylor series of the function $f(z)=\frac{z+1}{z-1}$ centered at 0 and determine its radius of convergence. Justify your answer.
b) Find the Laurent series of the function $f(z)$, given in part a), valid in the domain $|z|>1$.
5. (12 points) a) Use the definition of contour integrals, in order to prove the equality

$$
\begin{equation*}
\int_{C} e^{\bar{z}} d z=\int_{C} e^{4 / z} d z \tag{2}
\end{equation*}
$$

where $C$ is the circle $\{|z|=2\}$, traversed counterclockwise.
Caution: The exponent of the integrand, on the left hand side, is the complex conjugate $\bar{z}$ of $z$.
b) Find the Laurent series of $e^{4 / z}$ centered at zero and classify the type of singularity at $z=0$.
c) Use the equality (2) in order to evaluate the integral $\int_{C} e^{\bar{z}} d z$.
6. (12 points) Evaluate the integral

$$
\int_{0}^{2 \pi} \frac{d \theta}{5+4 \sin (\theta)}
$$

7. (12 points) Let $C_{A}$ be the straight line segment from $A+i A$ to $-A+i A$, where $A$ is a positive real number and $A>1$. Prove the inequality

$$
\left|\int_{C_{A}} \frac{e^{i z}}{z^{2}+1} d z\right| \leq \frac{2 A e^{-A}}{A^{2}-1}
$$

8. (12 points) Determine whether the following statements are true or false. Justify your answers!
a) If $f(z)$ and $g(z)$ are analytic at a point $z_{0}$ and $g\left(z_{0}\right)=g^{\prime}\left(z_{0}\right)=0$, but both $f\left(z_{0}\right)$ and $g^{\prime \prime}\left(z_{0}\right)$ are non-zero, then

$$
\operatorname{Res}_{z=z_{0}}\left(\frac{f}{g}\right)=0
$$

b) There exists an entire non-constant function $f(z)$ satisfying the inequality

$$
|f(z)| \leq|z| e^{-|z|}
$$

c) If $C$ is a simple closed contour, and $z_{0}$ does not belong to the domain $D$ bounded by $C$, then there is a single valued branch of $\log \left(z-z_{0}\right)$, defined for all $z$ in $D$.
d) There exists an entire function, whose real part is $x e^{y}$.
9. (12 points) Evaluate the improper integral $\int_{0}^{\infty} \frac{d x}{x^{4}+1}$. Simplify your answer as much as possible. Carefully state any theorem you use.

