Math 421 Final Exam Fall 2002

Name: My solution

Solve problem 1 and 7 out of problems 2 to 9. If you solve all 9, then problem 9 will not be graded. Please fill in: Please do not grade Problem number _____.

3 pts

1. (16 points) a) Show that the Laurent series of $\frac{1}{\sin(z)}$, centered at 0, has the form

$$\frac{1}{\sin(z)} = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \cdots \text{terms of order at least five.} \tag{1}$$

(You can use equality (1) in the subsequent parts, even if you do not derive it).

$$\left(\frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + 0z^4\right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) =$$

$$1 + z^2\left(\frac{1}{6} - \frac{1}{6}\right) + z^4\underbrace{\left(\frac{1}{5!} \underbrace{- \frac{1}{36} + \frac{7}{360}}_{-\frac{3}{360} = -\frac{1}{120}}\right)}_{= 0} +$$

4 pts

b) Find the principal part at $z = 0$ of the function $f(z) = \dfrac{1-z}{z^5 \cdot \sin(z)}$

$$\frac{(1-z)}{z^5}\left[\frac{1}{z^6} + \frac{1}{6}z^{-4} + \frac{7}{360}z^{-2} + \text{terms of order} \geq 0\right]$$

$$= \left[\frac{1}{z^6} - \frac{1}{z^5} + \frac{1}{6}(z^{-4} - z^{-3}) + \frac{7}{360}(z^{-6} - z^{-2}) + \cdots\right.$$

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given in part b

4 pts c) Find all the singularities of $f(z)$ in the disk $\{|z| < 4\}$ and determine their type (isolated, removable, pole of what order, essential).

$$f(z) = \frac{1-z}{z^5 \cdot \sin(z)}$$

0 is a pole of order 6, 2 pts

2 pts $\sin(z) = 0$ at $z = \pi$ and $z = -\pi$. It is a zero of order 1 because $\cos'(\pm\pi) = -1 \neq 0$

so, $\pi, -\pi$ are simple poles

5 pts d) Find the residue at each isolated singularity in D.

2 pts $\mathrm{Res}_{z=0} f(z) = -\frac{7}{360}$ from part a.

value of $\frac{1-z}{\frac{\partial}{\partial z}(z^5 \sin(z))}$ at $z=\pi$

2 pts $\mathrm{Res}_{z=\pi} f(z) = \frac{1-\pi}{\pi^5 \cdot \cos(\pi)} = \frac{\pi - 1}{\pi^5}$

1 pt $\mathrm{Res}_{z=-\pi} f(z) = \frac{1-(-\pi)}{(-\pi)^5 \cos(-\pi)} = +\frac{1+\pi}{\pi^5}$

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6 pts

2. (12 points) a) Compute $\sin(\frac{\pi}{4}+i\ln(3))$. Simplify your answer as much as possible.

$$\sin(z)=\frac{e^{iz}-e^{-iz}}{2i}=\frac{1}{2i}\Big[\underbrace{e^{-\ln(3)+\frac{\pi}{4}i}}_{=\frac{1}{3\sqrt{2}}(1+i)}-e^{\ln(3)-\frac{\pi}{4}i}\Big]=$$

2 pts

$$\frac{1}{3\sqrt{2}}(1+i)-\frac{3}{\sqrt{2}}(1-i)$$

3 pt

$$=\frac{1}{2i}\Big[\underbrace{\left(\frac{1}{3\sqrt{2}}-\frac{3}{\sqrt{2}}\right)}_{\frac{-8}{3\sqrt{2}}}+i\underbrace{\left(\frac{1}{3\sqrt{2}}+\frac{3}{\sqrt{2}}\right)}_{\frac{10}{3\sqrt{2}}}\Big]=\underbrace{\left[\left(\frac{5}{3\sqrt{2}}\right)+i\left(\frac{4}{3\sqrt{2}}\right)\right]}_{1\text{ pt}}$$

(6 points) b) Find all solutions of the equation $\cos(z)=i$.

$$w=e^{iz}\qquad \frac{e^{iz}+e^{-iz}}{2}=i$$

1 pt

$$\frac{w+\frac{1}{w}}{2}=i$$

2 pts

$$w^2-2iw+1=0$$

$$w=\frac{i\pm\sqrt{-1-1}}{1}=i(1\pm\sqrt{2})$$

$$w_1=(1+\sqrt{2})i \qquad w_2=-(\sqrt{2}-1)i$$

1 pts

$$z=\frac{\ln(1+\sqrt{2})+\left(\frac{\pi}{2}+2\pi n\right)i}{i}\quad n\text{ integer}$$

or

$$z=\frac{\ln(1-\sqrt{2})+\left(-\frac{\pi}{2}+2n\pi\right)i}{i}\quad n\text{ integer}$$

2 pts

3. (12 points) Compute the integral $\int_C \underbrace{\frac{\sin(z)+1}{e^{3z}-e^z}}_{f(z)} dz$, where C is the circle $\{|z|=1\}$ traversed counterclockwise.

$$\frac{(\sin(z)+1)}{e^{z}} \cdot \frac{1}{e^{2z}-1}$$

$$2\pi i \cdot \operatorname{Res}_{z=0} f(z) = 2\pi i \left(\frac{1}{1} \cdot \frac{1}{\underbrace{\frac{\partial}{\partial z}\Big|_{z=0} (e^{2z}-1)}_{2e^0}} \right) = 2\pi i \cdot \frac{1}{2} = \pi i.$$

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6 pts

4. (12 points) a) Find the Taylor series of the function $f(z) = \dfrac{z+1}{z-1}$ centered at 0 and determine its radius of convergence. Justify your answer.

$$\frac{z+1}{z-1} = 1 + \frac{2}{z-1} = 1-(2)\frac{1}{1-z} = 1-2(1+z+z^2+\dots)$$

$$= -1 + \sum_{n=1}^{\infty}(-2)z^n$$

Grading:

6 pts

b) Find the Laurent series of the function $f(z)$, given in part a), valid in the domain $|z| > 1$.

$$\frac{z+1}{z-1} = \frac{1+\frac{1}{z}}{1-\frac{1}{z}} = -\frac{-1-\frac{1}{z}}{1-\frac{1}{z}} = -1 + \frac{2}{1-\frac{1}{z}} = -1 + 2\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n =$$

$$= 1 + \sum_{n=1}^{\infty} 2\left(\frac{1}{z}\right)^n$$

Grading:

Name:______________________

5. (12 points) a) Use the definition of contour integrals, in order to prove the equality

$$\int_C e^{\bar{z}}dz = \int_C e^{4/z}dz, \qquad (2)$$

where C is the circle $\{|z| = 2\}$, traversed counterclockwise.

4 pts

Caution: The exponent of the integrand, on the left hand side, is the complex conjugate $\bar{z}$ of z.

$z(t) = 2e^{i\theta}, \quad 0 \le \theta \le 2\pi$

$$\int_C e^{\bar{z}}dz = \int_0^{2\pi} e^{2e^{-i\theta}} \cdot 2i\, e^{i\theta}\, d\theta$$

$$\int_C e^{4/z}dz = \int_0^{2\pi} e^{2e^{-i\theta}} \cdot 2i e^{i\theta}\, d\theta$$

4 pts

b) Find the Laurent series of $e^{4/z}$ centered at zero and classify the type of singularity at $z = 0$.

$$e^{\frac{4}{z}} = 1 + \frac{4}{z} + \frac{4^2}{2} z^{-2} + \cdots + \frac{1}{n!}\frac{4^n}{z^n} + \cdots$$

Essential singularity.

4 pts

c) Use the equality (2) in order to evaluate the integral $\int_C e^{\bar{z}}dz$.

$$\int_C e^{\bar{z}}dz = \int_C e^{4/z}dz = 2\pi i \cdot \operatorname{Res}_{z=0} e^{\frac{4}{z}} = (2\pi i) \cdot 4 = 8\pi i$$

6. (12 points) Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin(\theta)}.$$

$z = e^{i\theta}$ $\quad dz = ie^{i\theta}d\theta$ $\quad \boxed{d\theta = \frac{dz}{iz}}$

$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \boxed{\frac{z - \frac{1}{z}}{2i}}$

$$\int_C \frac{dz}{iz\left(5 + \frac{4}{2}\left[\frac{z - \frac{1}{z}}{2i}\right]\right)} =$$

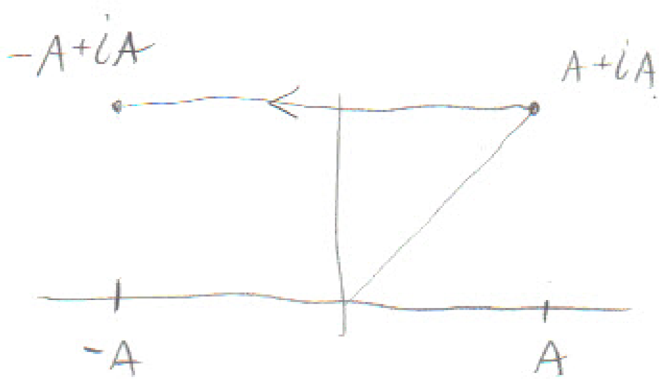
$$\int_C \frac{dz}{5iz + 2z^2 - 2} =$$

$$\frac{-5i \pm \sqrt{\underbrace{-25+16}_{-9}}}{4} = \frac{-8i}{4}, \boxed{\frac{-i}{2}}$$

$$(2\pi i)\operatorname{Res}_{z=\frac{-i}{2}}\left(\frac{1}{2z^2+5iz-2}\right) = (2\pi i)\frac{1}{\underbrace{(4z+5i)}_{3i}}\Big|_{z=\frac{-i}{2}} = \frac{2\pi}{3}$$

7. (12 points) Let C_A be the straight line segment from $A + iA$ to $-A + iA$, where A is a positive real number. Prove the inequality

$$\left| \int_{C_A} \frac{\cancel{z}e^{iz}}{z^2+1} dz \right| \le \frac{2Ae^{-A}}{A^2-1}.$$



$$z(t) = (A+iA) + (-2A)t$$
$$0 \le t \le 1$$

$$\left|\frac{1}{z^2+1}\right| \le \frac{1}{|z|^2-1} \le \frac{1}{A^2-1}$$ 3 pt

$$\left|e^{iz(t)}\right| = \left|e^{-A}\right| \cdot \underbrace{\left|e^{iA(1-2t)}\right|}_{1}$$ 3 pt

$$\left|\int_{C_A} \frac{e^{iz}}{z^2+1} dz\right| \le \int | \quad | \, |dz| \le \frac{e^{-A}}{A^2-1} \overbrace{\text{length}(C_A)}^{2A} = \frac{2Ae^{-A}}{A^2-1}$$

Grading:

8. (12 points) Determine whether the following statements are true or false. Justify your answers!

a) If $f(z)$ and $g(z)$ are analytic at a point z_0 and $g(z_0) = g'(z_0) = 0$, but both $f(z_0)$ and $g''(z_0)$ are non-zero, then

$$\mathrm{Res}_{z=z_0}\left(\frac{f}{g}\right) = 0.$$

False

$$\frac{e^z}{z^2}$$

b) There exists an non-constant entire function $f(z)$ satisfying the inequalities

$$e^{-|z|} \leq |f(z)| \leq |z|e^{-|z|}$$

False.

Use Liouville's Thm. $\lim_{z\to\infty} |z|e^{-|z|} = 0$

Thus, f is bounded.

c) If C is a simple closed contour, and z_0 does not belong to the domain D bounded by C, then there is a single valued branch of $\log(z - z_0)$, defined for all z in D.

True, $\frac{1}{z-z_0}$ is analytic on D.

d) There exists an entire function, whose real part is xe^y.

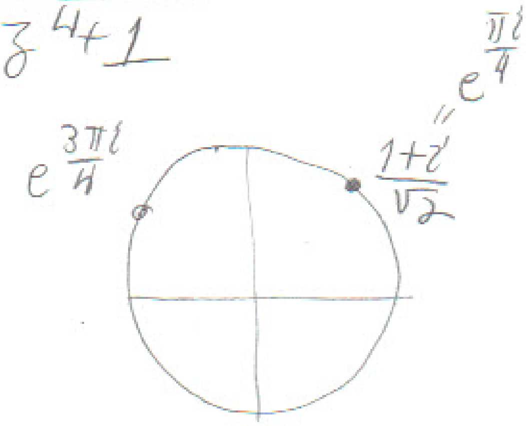
u

False, u is not harmonic

9. (12 points) Evaluate the improper integral

$$I = \int_0^\infty \frac{dx}{x^4+1}$$

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{2\pi i}{2} \sum_{\substack{z_k \text{ in} \\ \text{upper} \\ \text{half} \\ \text{plane}}} \operatorname{Res}_{z=z_k} \frac{1}{z^4+1}$$



$$= \pi i \cdot \left[\frac{1}{4(e^{\frac{\pi i}{4}})^3} + \frac{1}{4(e^{\frac{3\pi i}{4}})^3} \right]$$

$$= \frac{\pi}{4} i \left[e^{-\frac{3\pi i}{4}} + \underset{e^{-\frac{\pi i}{4}}}{\underset{\parallel}{e^{-\frac{9\pi i}{4}}}} \right] = \frac{\pi}{4}\left[\underbrace{e^{-\frac{\pi i}{4}} + e^{\frac{\pi i}{4}}}_{\frac{2}{\sqrt{2}}} \right] = \frac{\pi}{2\sqrt{2}}$$