Math 421 Final Exam Spring 2012

**Show all your work and justify al your answers!**

1. (12 points) Determine the number of zeroes, counting multiplicities, of the equation $z^7 - 4z^3 + z - 1 = 0$ in the annulus $\{z \ : \ 1 < |z| < 2\}$. State carefully any theorem you use.

2. (14 points)

   (a) Let $k$ and $m$ be two non-negative integers and $\phi(z)$ a function, which is analytic at $z_0$, with $\phi(z_0) \neq 0$. Set $g(z) = (z - z_0)^m \phi(z)$. Calculate the residue $\mathrm{Res}_{z=z_0}\left(z^k \frac{g'(z)}{g(z)}\right)$. Justify your answer!

   (b) Let $f$ be an analytic function on a simply connected open set $U$ and $C$ a positively oriented simple closed contour in $U$, not passing through any zero of $f$. Suppose that $f$ has $n$ zeroes $z_1, \ldots, z_n$ in the domain bounded by $C$, and let $m_j$ denote the multiplicity of $z_j$ as a zero of $f$. Prove the following equality for every non-negative integer $k$.

$$\int_C \frac{z^k f'(z)}{f(z)} dz \quad = \quad 2\pi i \sum_{j=1}^{n} m_j z_j^k.$$

3. (12 points) Let $C$ be the unit circle oriented counterclockwise. Prove the following equalities.

$$\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2(\theta)} \quad = \quad \int_C \frac{4zi}{z^4 - 6z^2 + 1} dz \quad = \quad \sqrt{2}\pi.$$

4. (12 points) Prove the following equality for all positive real numbers $a$ and $b$.

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(x^2 + b^2)^2} dx \quad = \quad \frac{\pi}{2b^3}(1 + ab)e^{-ab}.$$

5. (12 points)

   (a) Find the Laurent series of $e^{1/z}$ valid in $\{z \ : \ |z| > 0\}$.

   (b) Let $C$ be the unit circle, oriented counterclockwise. Evaluate the following integral.

$$\frac{1}{2\pi i} \int_C \left(\frac{1}{z^2} + z + z^3\right) e^{1/z} dz.$$

6. (14 points) Let $f(z) = \dfrac{3}{(z-1)(z+2)}$.

   (a) Find the Taylor series, centered at the origin, of the function $f$. Where is $f(z)$ equal to the sum of its Taylor series? Justify your answer.

   (b) Find the Laurent series representing $f$ in the annulus $\{z \ : \ 1 < |z| < 2\}$.

   (c) Find the Laurent series representing $f$ in the domain $\{z \ : \ |z| > 2\}$.

7. (12 points) Let $S_r$ be the upper half of the circle of radius $r$ centered at 0, and parametrized by $z(\theta) = re^{i\theta}$, $0 \leq \theta \leq \pi$.

   (a) Let $g$ be a function analytic in some open disk centered at 0. Prove that $\lim_{r \to 0} \int_{S_r} g(z)dz = 0$.

   (b) Let $f(z)$ be a meromorphic function with a simple pole at 0. Set $B := \mathrm{Res}_{z=0}(f)$. Prove that $\lim_{r \to 0} \int_{S_r} f(z)dz = \pi i B$. Hint: Consider the function $g(z) = f(z) - \frac{B}{z}$.

8. (12 points) Let $C$ be the circle of radius 2 centered at the origin and oriented counterclockwise. Evaluate the integral $\frac{1}{2\pi i} \int_C \frac{Log(z+3)}{(z-1)^{n+1}} dz$, for all integers $n \geq 0$.