Math 421 Midterm 2 Spring 2012

Show all your work and justify al your answers!

1. (18 points) Let C be the circle of radius 2 centered at the origin and oriented counterclockwise. Evaluate the following integrals.

   (a) $\displaystyle\int_C \frac{dz}{z^2+2z-3}$

   (b) $\displaystyle\int_C Log(z+5)dz$, where $Log(z)$ is the principal branch of the logarithm function with argument in $(-\pi, \pi)$.

2. (18 points) Let C be the unit circle oriented counterclockwise and let z_0 be a complex number satisfying $|z_0| \neq 1$. Prove the equality

$$\int_C \frac{\sin(z^2)}{(z-z_0)^2}dz \quad = \quad \int_C \frac{2z\cos(z^2)}{z-z_0}dz.$$

3. (10 points) Let $f(z) = e^{(z^2)}\sin(z^4+z-2)$. Does f have an anti-derivative? In other words, does there exist an entire function $F(z)$, such that $F'(z) = f(z)$. Carefully justify your answer.

4. (18 points) Let C_1 be the circle of radius 2 centered at $2i$ oriented counterclockwise. Let C_2 be the circle of radius 5 centered at the origin oriented counterclockwise. Set $f(z) := \dfrac{1}{(z^2+1)^2}$. Evaluate the difference $\displaystyle\int_{C_2} f(z)dz - \int_{C_1} f(z)dz$.
   Hint: Cauchy-Goursat's Theorem for multiply connected regions helps. Clearly state it and explain why its all hypothesis are satisfied in the set-up in which you apply it..

5. (18 points)

   (a) Let U be the upper half-plane $\{x+iy \ : \ y>0\}$ of the complex plane. Set $g(z) := e^{iz}$. Describe geometrically the image $g(U)$ of U under the function g.

   (b) Suppose that $f(z)$ is an entire function. Write $f(x+iy) = u(x,y)+iv(x,y)$. Assume that $v(x,y) \geq u(x,y)$, for all points (x,y) in the plane. Note that the assumption means that the values of f are all in the half-plane above the line $v=u$ in the (u,v) plane. Show that $f(z)$ is a constant function.
   Hint: Consider the function $g(z) = e^{\lambda f(z)}$, for a suitable constant λ.

6. (18 points) Let C_R denote the circle of radius R, $R>2$, centered at the origin and oriented counterclockwise. Set $I_R \ := \displaystyle\int_{C_R} \frac{z^2+9}{z^4+3z^2+2}dz$.

   (a) Prove the inequality

$$|I_R| \leq \frac{2\pi R(R^2+9)}{(R^2-1)(R^2-2)}. \tag{1}$$

   (b) Prove that $\lim_{R\to\infty} I_R = 0$. Note that you are taking the limit of the **left** hand side of equation (1).

   (c) Use part 6b to prove that $I_R = 0$, for all $R \geq 2$.