Math 421 Final Exam Spring 2012

Name: My Solution

Show all your work and justify al your answers!

1. (12 points) Determine the number of zeroes, counting multiplicities, of the equation $z^7 - 4z^3 + z - 1 = 0$ in the annulus $\{z : 1 < |z| < 2\}$. State carefully any theorem you use.

Rouche's Theorem: Let C be a simple closed contour and f, g analytic along C and throughout the domain U interior to C. Assume that

$$|f(z) - g(z)| < |f(z)|$$

for all $z \in C$. Then f and g have the same number of zeroes in U, when counted with multiplicities. Furthermore, neither f, nor g, have zeroes in C.

Set $g(z) = z^7 - 4z^3 + z - 1$.

Zeroes of g in the unit disk:

Set $f(z) = -4z^3$. Then f has 3 zeroes in the unit disk. If $|z|=1$, then

$$|f(z) - g(z)| = |z^7 + z - 1| \le |z|^7 + |z| + 1 \underset{|z|=1}{\le} 3 < 4 = |f(z)|.$$

Thus g has 3 roots in the open unit disk and none along the unit circle.

Zeroes of g in the disk $\{z : |z| < 2\}$.

Set $f(z) = z^7$. If $|z| = 2$, then

$$|f(z) - g(z)| = |4z^3 - z + 1| \le 4|z|^3 + |z| + 1 = 32 + 2 + 1 = 35 < 128 = |z|^7.$$

Thus g has 7 zeroes, counted with multiplicities in the disk of radius 2.

Conclusion: g has $7 - 3 = 4$ zeroes in the annulus.

2. (14 points)

(a) Let k and m be two non-negative integers and $\phi(z)$ a function, which is analytic at z_0, with $\phi(z_0) \neq 0$. Set $g(z) = (z - z_0)^m \phi(z)$. Calculate the residue $\mathrm{Res}_{z=z_0}\left(z^k \frac{g'(z)}{g(z)}\right)$. Justify your answer!

$$z^k \cdot \frac{g'(z)}{g(z)} = z^k\left(\frac{m}{z-z_0} + \frac{\phi'(z)}{\phi(z)}\right)$$

$\mathrm{Res}_{z_0}\left(z^k \frac{\phi'(z)}{\phi(z)}\right) = 0$, since the function is analytic at z_0.

[If $z_0 = 0$ and $k \geq 0$, then $z^k \cdot \frac{m}{z-z_0}$ is analytic (removable sing), so $\mathrm{Res}_{z_0}\left(z^k \cdot \frac{m}{z-z_0}\right) = 0$.

[If $z_0 \neq 0$, then z_0 is a simple pole of $z^k \frac{m}{z-z_0}$ and $\mathrm{Res}_{z_0}\left(z^k \frac{m}{z-z_0}\right) = z_0^k \cdot m$. Thus $\boxed{\mathrm{Res}_{z_0}\left(z^k \frac{\phi'(z)}{\phi(z)}\right) = z_0^k \cdot m}$

(b) Let f be an analytic function on a simply connected open set U and C a positively oriented simple closed contour in U, not passing through any zero of f. Suppose that f has n zeroes $z_1, \ldots, z_n$ in the domain bounded by C, and let m_j denote the multiplicity of z_j as a zero of f. Prove the following equality for every non-negative integer k.

$$\int_C \frac{z^k f'(z)}{f(z)} dz = 2\pi i \sum_{j=1}^{n} m_j z_j^k.$$

Set $\phi_i(z) := f(z)/(z-z_i)^{m_i}$

Then ϕ_i has a removable singularity at z_i and $\phi_i(z_i) \neq 0$, and $f(z) = (z-z_i)^{m_i}\phi_i(z)$. By part a, we have $\mathrm{Res}_{z_i}\left(\frac{z^k f'(z)}{f(z)}\right) = z_i^k m_i$

Cauchy's residue Formula yields

$$\int_C \frac{z^k f'(z)}{f(z)} dz \overset{\text{4 pts}}{=} 2\pi i \sum_{j=1}^{m} \mathrm{Res}_{z=z_j}\left(\frac{z^k f'(z)}{f(z)}\right) =$$

$$= 2\pi i \sum_{j=1}^{m} z_j^k m_j.$$

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3. (12 points) Let C be the unit circle oriented counterclockwise. Prove the following equalities.

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2(\theta)} = \int_C \frac{4zi}{z^4-6z^2+1}dz = \sqrt{2}\pi.$$

$$\int_{-\pi}^{\pi} \frac{1}{1+\sin^2(\theta)}\, d\theta = \int_C \frac{1}{1+\left(\frac{z-\frac{1}{z}}{2i}\right)^2} \cdot \frac{1}{iz}\, dz =$$

2 pts for each

$$\begin{cases} z = e^{i\theta} \\ d\theta = \frac{1}{iz} dz \\ \sin(\theta) = \frac{z-\frac{1}{z}}{2i} \end{cases}$$

$$\frac{1}{iz} = \frac{iz}{(iz)^2}$$

$$\int_C \frac{iz}{-z^2 + \frac{(z^2-1)^2}{4}}\, dz = \int_C \frac{4iz}{z^4-6z^2+1}\, dz =: \underline{I}$$

Set $w = z^2$. Roots of $w^2-6w+1=0$ are

$$w_{1,2} = +3 \pm \sqrt{9-1} = 3 \pm \sqrt{8}$$

zeroes of z^4-6z^2+1 in the open unit disk:

$z_{1,2} = \pm\sqrt{3-\sqrt{8}}$. Each is a simple pole.

$$\operatorname{Res}_{z=z_j} \frac{4iz}{z^4-6z^2+1} = \frac{4iz}{4z^3-12z}\Big|_{z=z_j} = i\,\frac{1}{z^2-3}\Big|_{z_j} = i\,\frac{1}{z_j^2-3} =$$

$$= i\,\frac{1}{(3-\sqrt{8})-3} = \boxed{-\frac{i}{2\sqrt{2}}},\quad j=1,2.$$

Thus, $I = 2\pi i\left(\operatorname{Res}_{z=z_1}\left(\frac{4iz}{z^4-6z^2+1}\right) + \operatorname{Res}_{z=z_2}(\;=\;)\right) =$

$$= 2\pi i\left(-\frac{\cancel{2}i}{\cancel{2}\sqrt{2}}\right) = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi.$$

4. (12 points) Prove the following equality for all positive real numbers a and b.

$$I = \int_{-\infty}^{\infty} \frac{\cos(ax)}{(x^2+b^2)^2}dx = \frac{\pi}{2b^3}(1+ab)e^{-ab}.$$

$$\text{" } \mathrm{Re}\, \frac{e^{iax}}{(x^2+b^2)^2}$$

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$$I = \mathrm{Re}\left\{ 2\pi i \left(\sum_{\substack{z_k \text{ pole} \\ \text{in upper half plane}}} \mathrm{Res}_{z=z_k} \frac{e^{iaz}}{(z^2+b^2)^2} \right) \right\}$$

$z_1 = ib$, $z_2 = -ib$, Not in the upper half plane

$$\mathrm{Res}_{z_1=ib} \frac{e^{iaz}}{\underbrace{(z^2+b^2)^2}_{=(z-ib)^2(z+ib)^2}} = \frac{\partial}{\partial z}\Big|_{z=ib} \left(\frac{e^{iaz}}{(z+ib)^2} \right) = \frac{iae^{iaz}(z+ib) - 2e^{iaz}\cancel{(z+ib)}}{(z+ib)^{\cancel{4}\,3}}\Bigg|_{z=ib}$$

$$= e^{-ab}\left(\frac{\overbrace{ia(2ib)}^{-2ab} - 2}{-i8b^3} \right) = \frac{1}{4}e^{-ab}\left(\frac{1+ab}{ib^3} \right)$$

$$\text{So } I = \mathrm{Re}\left\{ \frac{\cancel{2}\pi \cancel{i}}{\cancel{4}_{2} \cancel{i} b^3} e^{-ab}(1+ab) \right\} = \frac{\pi}{2b^3}e^{-ab}(1+ab).$$

5. (12 points)

6

(a) Find the Laurent series of $e^{1/z}$ valid in $\{z : |z| > 0\}$.

$$e^z = 1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} + \cdots$$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \cdots + \frac{1}{n!}\frac{1}{z^n} + \cdots$$

(b) Let C be the unit circle, oriented counterclockwise. Evaluate the following integral.

$$I := \frac{1}{2\pi i}\int_C \underbrace{\left(\frac{1}{z^2} + z + z^3\right) e^{1/z}}_{f(z)} dz.$$

The function $f(z)$ is analytic in $\{z ; |z| > 0\}$

(2 pts)

$I = \operatorname{Res}_{z=0} f(z) = $ coef of $\frac{1}{z}$ in the Laurent series of f

$\uparrow$

Cauchy-Residue Formula

$= \left(\text{coef of } z\right) + \left(\text{coeff of } \frac{1}{z^2}\right) + \left(\text{coeff of } \frac{1}{z^4}\right)$

in the Laurent series of $e^{1/z}$

$$= 0 + \frac{1}{2} + \frac{1}{4!} = \frac{12+1}{24} = \boxed{\frac{13}{24}}$$

6. (14 points) Let $f(z) = \frac{3}{(z-1)(z+2)}$. $= \frac{\overset{1}{\overset{\shortparallel}{A}}}{z-1} + \frac{\overset{-1}{\overset{\shortparallel}{B}}}{z+2} = \frac{A(z+2)+B(z-1)}{(\ )(\ )}$

(a) Find the Taylor series, centered at the origin, of the function f. Where is $f(z)$ equal to the sum of its Taylor series? Justify your answer.

$$\frac{1}{z-1} = \frac{-1}{1-z} = -(1+z+z^2+\dots+z^n)$$

$$\frac{-1}{z+2} = \left(\frac{-1}{2}\right)\frac{1}{1-\left(-\frac{z}{2}\right)} = \left(-\frac{1}{2}\right)\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n z^n = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1} z^n$$

$$f(z) = \sum_{n=0}^{\infty}\left[-1+\left(-\frac{1}{2}\right)^{n+1}\right] z^n$$

In $\{z : |z|<1\}$, by Taylor's Theorem, since this is the largest open disk centered at 0, on which f is analytic,
At points with $|z|=1$, the series diverges by the n-term test.

(b) Find the Laurent series representing f in the annulus $\{z : 1 < |z| < 2\}$.

$$\frac{1}{z-1} = \left(\frac{1}{z}\right)\frac{1}{1-\left(\frac{1}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=1}^{\infty}\frac{1}{z^n}$$

$$f(z) = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1} z^n +$$

(c) Find the Laurent series representing f in the domain $\{z : |z| > 2\}$.

$$\frac{-1}{z+2} = \left(\frac{-1}{z}\right)\frac{1}{1+\frac{2}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}(-2)^n\frac{1}{z^n} = \sum_{n=0}^{\infty}-(-2)^n\frac{1}{z^{n+1}} = \sum_{k=1}^{\infty}-(-2)^{k-1}\frac{1}{z^k}$$

$$f(z) = \sum_{n=1}^{\infty}\left[1-(-2)^{n-1}\right]\frac{1}{z^n}$$

7. (12 points) Let S_r be the upper half of the circle of radius r centered at 0, and parametrized by $z(\theta) = re^{i\theta}$, $0 \le \theta \le \pi$.

(a) Let g be a function analytic in some open disk centered at 0. Prove that $\lim_{r\to 0} \int_{S_r} g(z)dz = 0$.

Let $R>0$ such that g is analytic in $\overline{D}_R = \{z : |z| \le R\}$. Let M be $\max\{|g(z)| : |z| \le R\}$. M exists and is finite, since g is continuous on $\overline{D}_R$ and $\overline{D}_R$ is closed and bounded.

Then $\left|\int_{S_r} g(z)\,dz\right| \le M \text{ legth}(S_r) = M \cdot 2\pi r \xrightarrow{r\to 0} 0$.

(b) Let $f(z)$ be a meromorphic function with a simple pole at 0. Set $B := \mathrm{Res}_{z=0}(f)$. Prove that $\lim_{r\to 0} \int_{S_r} f(z)dz = \pi i B$. Hint: Consider the function $g(z) = f(z) - \frac{B}{z}$.

The principal part of f at 0 is $\frac{B}{z}$, by def of Residue and the fact that 0 is a simple pole, Hence, the principal part of $f(z) - \frac{B}{z}$ is zero and so $z_0 = 0$ is a removable singularity of $f(z) - \frac{B}{z}$. Hence, $f(z) - \frac{B}{z}$ extends to a function $g(z)$ defined and analytic at 0.

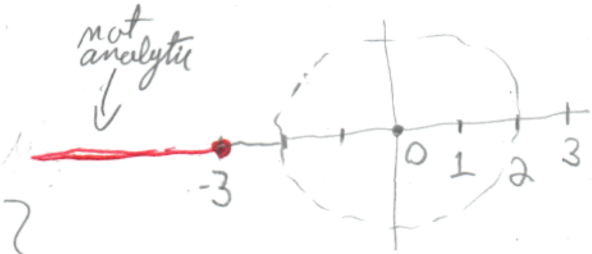
$$\lim_{r\to 0} \int_{S_r} f(z)\,dz = \lim_{r\to 0} \int_{S_r} \left(g(z) + \frac{B}{z}\right) dz =$$

$$= \lim_{r\to 0} \int_{S_r} g(z)\,dz + \lim_{r\to 0} \int_{S_r} \frac{B}{z}\,dz \overset{\text{part (a)}}{=}$$

$$= 0 + \lim_{r\to 0} \int_0^{\pi} \frac{B}{re^{i\theta}} \cdot rie^{i\theta}\,d\theta = \int_0^{\pi} Bi\,d\theta = \pi i B.$$

8. (12 points) Let C be the circle of radius 2 centered at the origin and oriented counterclockwise. Evaluate the integral $\frac{1}{2\pi i}\int_C \frac{Log(z+3)}{(z-1)^{n+1}}dz$, for all integers $n \geq 0$.

$= I_n$



$f(z) = Log(z+3)$ is analytic in $\mathbb{C} \setminus \{z = x + 0i : x \leq -3\}$.

Hence, it is analytic at all points of C and at all points interior to C.

Cauchy's Formula for higher derivatives apply and yields

$$f^{(n)}(1) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-1)^{n+1}}\,dz = n!\,I$$

So $I_n = \frac{1}{n!} f^{(n)}(1)$.

So $\boxed{I_0 = f(1) = Log(4) = \ln(4).}$ For $n \geq 1$,

$f'(z) = \frac{1}{z+3} = (z+3)^{-1}$

$f''(z) = (-1)(z+3)^{-2}$

$f^{(n)}(z) = (-1)^{n+1}(n-1)!\,(z+3)^{-n}$

$f^{(n)}(1) = \frac{(-1)^{n+1}}{4^n}(n-1)!$

$\frac{(-1)^{n+1}}{n\,4^n}$ //

$$\boxed{I_n = \frac{(-1)^{n+1}\cancel{(n-1)!}}{\cancel{n!}\,4^n}}, \quad n \geq 1.$$