

Name: My Solution

Solve problem 1 and only 7 out of problems 2 to 9. If you solve all 9, then problem 9 will not be graded. Please fill in: Please do not grade Problem number _____. Show all your work. Credit will **not** be given for an answer without a justification. Calculators may **not** be used in this exam.

1. (16 points) Given that the Taylor series of $\tan(z)$, centered at 0, has the form

$$\text{3 pto} \quad \tan(z) = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \dots \text{ terms of order at least seven.} \quad (1)$$

- a) Evaluate the fifth derivative $\tan^{(5)}(0)$ with as little calculations as possible.

$$\frac{\tan^{(5)}(0)}{5!} = \frac{2}{15}$$

$$\text{so} \quad \tan^{(5)}(0) = \frac{2 \cdot 5!}{15} = 2 \cdot 4 \cdot 2 = 16$$

- b) Find the principal part at $z = 0$ of the function $f(z) = \frac{(1+z)\tan(z)}{z^5}$

$$\frac{1}{z^5} \left(1+z \right) \left(z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \dots \right) = \underbrace{\frac{1}{z^4} + \frac{1}{z^3} + \left(\frac{1}{3} \right) \frac{1}{z^2} + \left(\frac{1}{3} \right) \frac{1}{z}}_{z+z^2+\frac{1}{3}z^3+\frac{1}{3}z^4+\frac{2}{15}z^5+\frac{2}{15}z^6+\dots} + \underbrace{\dots}_{\text{Principal Part}}$$

c) Find all the singularities of $f(z)$ (given in part b) in the disk $D = \{|z| < 4\}$ and determine their type (isolated, removable, pole of what order, essential).

A pole of order 4 at $z=0$ $\{$ +2

$$f(z) = \frac{(1+z) \sin(z)}{z^5 \cos(z)}$$

A simple pole at $z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$, $\{$ +2

4 pts

d) Find the residue at each isolated singularity in D .

$$\operatorname{Res}_{z=0} f(z) = \frac{1}{3} \quad \text{from part (a)}$$

$$\operatorname{Res}_{z=\frac{\pi}{2}} f(z) = \left[\frac{\left(1+\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right)}{\left(\frac{\pi}{2}\right)^5} \right] = - \frac{\left(1+\frac{\pi}{2}\right)}{\left(\frac{\pi}{2}\right)^5}$$

$\underbrace{\cos'\left(\frac{\pi}{2}\right)}$
 $\underbrace{-\sin'\left(\frac{\pi}{2}\right)}$

$$\operatorname{Res}_{z=-\frac{\pi}{2}} f(z) = \left[\frac{\left(1-\frac{\pi}{2}\right) \sin'\left(-\frac{\pi}{2}\right)}{\left(-\frac{\pi}{2}\right)^5} \right] = + \frac{\left(1-\frac{\pi}{2}\right)}{\left(\frac{\pi}{2}\right)^5}$$

$\underbrace{\cos'\left(-\frac{\pi}{2}\right)}$
 $\underbrace{-\sin'\left(-\frac{\pi}{2}\right)}$

2. (12 points) a) Compute $\sin(\pi + i \ln(3))$. Simplify your answer as much as possible.

$$\frac{e^{i[\pi + i \ln(3)]} - e^{-i[\pi + i \ln(3)]}}{2i} = \frac{e^{-\ln(3) + \pi i} - e^{\ln(3) - \pi i}}{2i} =$$

$$= (-1) \cdot \left[\frac{\frac{1}{3} - 3}{2i} \right] = \left(\frac{\frac{1}{3} - 3}{2} \right)i = -\frac{4}{3}i$$

8 points

b) Prove that all solutions of the equation $\cos(z) = 0$ are real and find all the solutions.

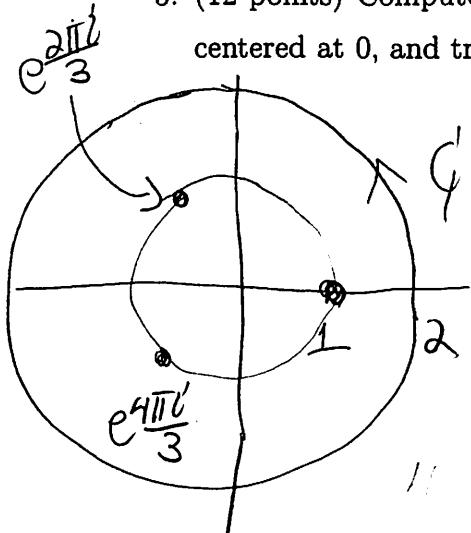
$$\frac{e^{iz} + e^{-iz}}{2} = 0$$

$$\frac{e^{iz} = -e^{-iz}}{e^{2iz} = -1} = e^{\pi i}$$

$$2iz = i[\pi + 2k\pi]$$

$$z = \frac{\pi}{2} + k\pi$$

3. (12 points) Compute the integral $\int_C \frac{z^5}{1-z^3} dz$, where C is the circle of radius 2, centered at 0, and traversed counterclockwise.



$$z^3 = 1$$

$$\int_C \frac{z^5}{1-z^3} dz = 2\pi i \cdot \left[\underbrace{\text{Res}_{z=1} \frac{z^5}{1-z^3}}_{\substack{\\ \frac{1}{-3(1^2)}}} + \underbrace{\text{Res}_{z=e^{2\pi i/3}} \frac{z^5}{1-z^3}}_{\substack{\\ \frac{(e^{2\pi i/3})^5}{-3(e^{2\pi i/3})^2}}} + \underbrace{\text{Res}_{z=e^{4\pi i/3}} \frac{z^5}{1-z^3}}_{\substack{\\ \frac{(e^{4\pi i/3})^5}{-3(e^{4\pi i/3})^2}}} \right]$$

$$= 2\pi i [-1] = \boxed{-2\pi i}$$

- ~~7 pts~~
4. (12 points) a) Find the Taylor series of the function $f(z) = \frac{2z+1}{z^2+z-2} = \frac{1}{z-1} + \frac{1}{z+2}$ centered at 0 and determine its radius of convergence. Justify your answer.

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n =$$

$$\frac{1}{z+2} = \frac{1}{2} \frac{1}{1+\frac{z}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n 2^{-n} z^n = \sum_{n=0}^{\infty} (-1)^n 2^{-n-1} z^n$$

$$\beta(z) = \sum_{n=0}^{\infty} \left[-1 + (-1)^n 2^{-n-1} \right] z^n$$

Radius of conv = 1 = the radius of the largest disk centered at 0, which does not contain $1, -2$.

5 pts

- b) Find the Laurent series of the function $f(z)$, given in part a), valid in the annulus $1 < |z| < 2$.

$$\frac{1}{z-1} \text{ is analytic in } |z| > 1 \Rightarrow \frac{1}{z-1} = \frac{1}{z}, \quad \frac{1}{1-\frac{1}{z}} = \sum_{m=1}^{\infty} \left(\frac{1}{z}\right)^m$$

$$\frac{1}{z+2} \text{ is } " \text{ in } |z| < 2$$

$$\left\{ \sum_{m=-\infty}^{-1} z^m + \sum_{m=0}^{\infty} (-1)^m 2^{-m-1} z^m \right\}$$

5. (12 points) a) Use the definition of contour integrals to prove the equality

$$\int_C \sin(\bar{z}) dz = \int_C \sin(1/z) dz, \quad (2)$$

where C is the circle $\{z : |z| = 1\}$, traversed counterclockwise. Caution: The argument of the integrand, on the left hand side, is the complex conjugate \bar{z} of z .

$$\int_C \sin(\bar{z}) dz = \int_0^{2\pi} \sin(e^{-i\theta}) i e^{i\theta} d\theta$$

$\boxed{z = e^{i\theta}}$

$$\int_C \sin(1/z) dz = \int_0^{2\pi} \sin\left(\frac{1}{e^{i\theta}}\right) i e^{i\theta} d\theta$$

$\boxed{\frac{1}{e^{i\theta}} = \overline{e^{i\theta}}}$

2 pts

b) Find the Laurent series of $\sin(1/z)$ centered at zero and classify the type of singularity at $z = 0$.

$$\sin(\omega) = \omega - \frac{\omega^3}{3!} + \frac{\omega^5}{5!} + \dots + (-1)^n \frac{\omega^{2n+1}}{(2n+1)!} + \dots$$

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!} \frac{1}{z^3} + \frac{1}{5!} \frac{1}{z^5} + \dots + (-1)^n \frac{z^{-2n-1}}{(2n+1)!} + \dots$$

The origin is an ESSENTIAL singularity.

3 pts

c) Use the equality (2) in order to evaluate the integral $\int_C \sin(\bar{z}) dz$.

$$\int_C \sin(\bar{z}) dz = \int_G \sin\left(\frac{1}{z}\right) dz = 2\pi i \operatorname{Res}_{z=0} \sin\left(\frac{1}{z}\right) = (2\pi i) \cdot 1 = \boxed{2\pi i}$$

6. (12 points) Evaluate the integral

$$I = \int_0^{2\pi} \frac{d\theta}{2 + \cos(\theta)}.$$

Show all your work!

$$z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\frac{1}{z} = e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

$$\boxed{\cos(\theta) = \frac{(z + \frac{1}{z})}{2}}$$

$$dz = ie^{i\theta} d\theta = iz d\theta$$

$$\boxed{d\theta = \frac{dz}{iz}}$$

$$I = \int_{|z|=1} \frac{dz}{iz \left[z + \frac{z + \frac{1}{z}}{2} \right]} = -2i \int_{|z|=1} \frac{dz}{4z + z^2 + 1} = (-2i) 2\pi i \cdot \text{Res} \left(\frac{1}{z^2 + 4z + 1}, z = -2 + \sqrt{3} \right)$$

$$= 4\pi \cdot \frac{1}{2(-2+\sqrt{3})+4}$$

$$\frac{1}{\frac{d}{dz} (2z+4)} \Big|_{z=-2+\sqrt{3}}$$

$$= \frac{4\pi}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

7. (12 points) Let S_R be the upper-semi-circle of radius $R > 1$, given by the parametrization $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$. Prove the equality

$$\lim_{R \rightarrow \infty} \int_{S_R} \frac{z^2 dz}{1+z^4} = 0.$$

Hint: Find first an upper bound for the integral.

$$\left| \int_{S_R} \frac{z^2 dz}{1+z^4} \right| \leq \int_{S_R} \left| \frac{z^2}{1+z^4} \right| |dz| \leq \frac{R^2}{R^4 - 1}, \quad \pi R = \pi \frac{R^3}{R^4 - 1} \xrightarrow[R \rightarrow \infty]{} 0$$

(2 pts)
 $\frac{R^2}{R^4 - 1}$
 $\frac{R^2}{R^4 - 1}$

8. (12 points) Determine whether the following statements are true or false. Justify your answers!

- a) Let C be the circle $\{z : |z| = 1\}$, oriented counterclockwise. Assume that $f(z)$ is analytic in the punctured disk $0 < |z| < 2$, and the integrals $\int_C z^n f(z) dz$ vanish, for all integers $n \geq 0$. Then 0 is a removable singularity of f .

True,

The Laurent series of f in the given punctured disk is $\sum_{K=-\infty}^{\infty} c_K z^K$ where $c_K = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{K+1}} dz$.

Setting $K = -1$, we see that $c_K = 0$ for $K \leq -1$, and so the principal part of the Laurent series vanishes.

- b) There exists a function $F(z)$, analytic in the punctured unit disk $U = \{z : 0 < |z| < 1\}$, whose derivative $f(z) := F'(z)$ satisfies $\text{Res}_{z=0}(f(z)) = 1$.

False.

$$\text{Res}_{z=0} f(z) = \frac{1}{2\pi i} \int_C g(z) dz, \text{ where } G \text{ is a circle of}$$

radius r , $0 < r < 1$, centered at 0. The function f can not have an anti-derivative in U , since its contour integral along the closed curve G in U is non-zero.

- c) If f is a non-constant entire function and $|f(z)| \leq 2$, for every z on the unit circle $\{z : |z| = 1\}$, then f must map the unit disk $\{z : |z| < 1\}$ into the disk $\{z : |z| < 2\}$.

True, by the Maximum modulus principle,
 Since if f is non constant, and $|f(z)| \leq 2$, for $|z|=1$,
 then $|f(z)| < 2$, for all $|z| < 1$.

d) There exists an entire function, whose real part is e^{x+y} .
False.
 The real part of an analytic function is
 a Harmonic function.
 Set $u(x, y) = e^{x+y}$,
 Then $u_{xx} + u_{yy} = 2e^{x+y} \neq 0$,
 So u is not harmonic.

9. (12 points) Evaluate the improper integral

$$\int_0^\infty \frac{x^2}{x^4+1} dx.$$

$$\deg(x^2) = 2$$

$$\deg(x^4+1) = 4 \geq 2+2. \text{ So}$$

$$\int_0^\infty \frac{x^2}{x^4+1} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2}{x^4+1} dx = 2\pi i \sum_{\text{residues in upper-half plane}} \frac{\frac{z^2}{z^2}}{z^4+1} =$$

$$= 2\pi i \left[\underbrace{\text{Res}_{z=e^{\frac{\pi i}{4}}} \frac{\frac{z^2}{z^2}}{z^4+1}}_{z=e^{\frac{\pi i}{4}}} + \underbrace{\text{Res}_{z=e^{\frac{3\pi i}{4}}} \frac{\frac{z^2}{z^2}}{z^4+1}}_{z=e^{\frac{3\pi i}{4}}} \right]$$

$$\left(\frac{\frac{z^2}{z^2}}{4z^3} \right) \Big|_{z=e^{\frac{\pi i}{4}}} = \frac{e^{-\frac{\pi i}{4}}}{4} = \frac{1}{4} e^{-\frac{\pi i}{4}}$$

$$\left(\frac{\frac{z^2}{z^2}}{4z^3} \right) \Big|_{z=e^{\frac{3\pi i}{4}}} = \frac{e^{-\frac{3\pi i}{4}}}{4} = \frac{1}{4} e^{-\frac{3\pi i}{4}}$$

$$= \frac{\pi i}{4} \cdot \left[\underbrace{e^{-\frac{\pi i}{4}}}_{\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}} + \underbrace{e^{-\frac{3\pi i}{4}}}_{-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \frac{1}{\sqrt{2}}} \right] = \frac{\sqrt{2}\pi}{4} = \boxed{\frac{\pi}{2\sqrt{2}}}$$

$\underbrace{-\sqrt{2}i}$