8 pts

$= \ln|x+iy|$

1. (25 points) a) Prove that the function $u(x,y) = \dfrac{1}{2}\ln(x^2+y^2)$ is harmonic on $\mathbb{R}^2 \setminus \{0\}$ (on the whole plane minus the origin).

Method 1! u is the real part of the principal logarithm $\operatorname{Log}(x+iy) = \ln|x+iy| + i\operatorname{Arg}(x+iy)$, which is analytic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, hence it is harmonic.

(The assumption the its first and second partials exist and are continuous follows from the Corollary in Sec 52, and can also be verified directly.). u is also the real part of the branch of log with argument in $(0, 2\pi)$ analytic on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Hence it is harmonic on $\mathbb{C} \setminus \{0\}$.

Method 2; $u_x = \left(\frac{1}{2}\right)\dfrac{2x}{x^2+y^2} = \dfrac{x}{x^2+y^2}$, $u_y = \dfrac{y}{x^2+y^2}$

$u_{xx} = \dfrac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \dfrac{-x^2+y^2}{(x^2+y^2)^2}$ • $u_{yy} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$

$u_{xx} + u_{yy} = 0$ ✓ 1-st & 2-nd partials are continuous (as are u_{xy} and u_{yx}), on $\mathbb{R}^2 \setminus \{(0,0)\}$,

8 pts

b) Let v be a harmonic conjugate of the function u in part (a) in some open subset D of $\mathbb{C}\setminus\{0\}$. Show that for all $z = x+iy$ in D, the equality $u_x(x,y) + iv_x(x,y) = \dfrac{1}{z}$ holds. (You do **not** need to find v).

$u+iv$ is analytic on D, so the Cauchy-Riemann Equation hold.

Hence $v_x = -u_y = \dfrac{-y}{x^2+y^2}$, so

$u_x(x,y) + iv_x(x,y) = \dfrac{x-iy}{x^2+y^2} = \dfrac{\overline{x+iy}}{|x+iy|^2} = \dfrac{1}{x+iy}$

4 pts

c) Keep the notation of part b). Set $F(z) = u(x,y) + iv(x,y)$. Show that $F'(z) = \frac{1}{z}$.

F is analytic in D, by definition of harmonic conjugate, so F' exists and equal $F'(x+iy) \underset{\uparrow}{=} u_x(x,y) + i v_x(x,y) =$

C.R. Theorem (or by def of deriv limit)

$= \frac{1}{z}$.

↖ Part b

5 pts

d) Let u be the function in part (a). Is there a harmonic conjugate v of u defined on the whole of $\mathbb{C} \setminus \{0\}$? Find such a function v or show that it does not exist.

No, there does not exist a harmonic conjugate v of u defined on $\mathbb{C} \setminus \{0\}$. The proof is by contradiction. Assume there was. Then $F := u + iv$ is an antiderivative of $\frac{1}{z}$ on $\mathbb{C} \setminus \{0\}$, by part c. Hence,

$\int_C \frac{1}{z} dz = 0$ for every closed contour lying in $\mathbb{C} \setminus \{0\}$.

But if C is the unit circle $\{z(\theta) = e^{i\theta}; \ 0 \leq \theta \leq 2\pi\}$,

then $$\int_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} i e^{i\theta} d\theta = i \int_0^{2\pi} d\theta = 2\pi i \neq 0.$$

A contradiction.

2. (25 points) Let C be the straight line segment from z_0 to z_1. Parametrize C and use the definition of the contour integral to prove the equality

$$\int_C \bar{z}dz = \frac{1}{2}(z_1 - z_0)(\bar{z}_0 + \bar{z}_1).$$

Parametrization of C: $z(t) = z_0 + t(z_1 - \bar{z}_0)$, $0 \le t \le 1$.

12 pt

$$\int_C \bar{z}\,dz = \int_0^1 \underbrace{\overline{z_0 + t(z_1 - z_0)}}_{\bar{z}_0 + (\bar{z}_1 - \bar{z}_0)t} \frac{\partial}{\partial t} \overbrace{z(t)}^{(z_1 - z_0)} \, dt =$$

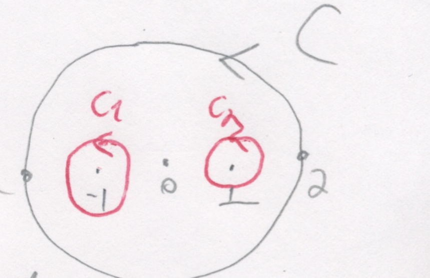
$$(z_1 - z_0)\,\bar{z}_0 \underbrace{\int_0^1 dt}_{1} + (\bar{z}_1 - \bar{z}_0)(z_1 - z_0) \underbrace{\int_0^1 t\, dt}_{\left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}} = (z_1 - z_0)\underbrace{\left[\bar{z}_0 + \frac{1}{2}(\bar{z}_1 - \bar{z}_0)\right]}_{\frac{1}{2}(\bar{z}_1 + \bar{z}_0)}.$$

3. (25 points) Compute the following integrals. Carefully justify your answer!

16 pts

(a) $\int_C \frac{2z}{z^2-1} dz$, where C is the circle of radius 2, centered at 0, oriented counterclockwise.

$I = \qquad \underbrace{}_{(z-1)(z+1)}$



Let C_1 be ~~the~~ the circle of radius $\frac{1}{2}$ centered at -1 positively oriented

Let C_2 be the circle of radius $\frac{1}{2}$ centered at 1 positively oriented.

Set $I_1 = \int_{C_1} \frac{2z}{z^2-1} dz$, $I_2 = \int_{C_2} \frac{2z}{z^2-1} dz$

$$I = \int_C \frac{2z}{z^2-1} dz \underset{\uparrow}{=} \int_{C_1} \frac{2z}{z^2-1} dz + \int_{C_2} \frac{2z}{z^2-1} dz = I_1 + I_2$$

Cauchy-Goursat for multiply connected domains, as $f(z) = \frac{2z}{z^2-1}$ is analytic at all points within and on C, but not within C_1 or within C_2.

$$I_1 = \int_{C_1} \frac{\left(\frac{2z}{z-1}\right)}{\underbrace{z+1}_{z-(-1)}} dz \underset{\uparrow}{=} (2\pi i) \frac{2(-1)}{-1-1} = 2\pi i$$

Cauchy's Integral Formula for the function $\frac{2z}{z-1}$

" " " " " , $\frac{2z}{z+1}$

$$I_2 = \int_{C_2} \frac{\left(\frac{2z}{z+1}\right)}{z-1} dz \overset{\downarrow}{=} (2\pi i) \frac{2 \cdot 1}{1+1} = 2\pi i,$$

So $\boxed{I = 4\pi i}$

9 pts

(b) $\int_C \frac{1}{\cos(z)} dz$, where C is the unit circle oriented counterclockwise.

$\frac{1}{\cos(z)}$ is analytic on $\mathbb{C}$ except at $z = \frac{\pi}{2} + k\pi$, k integer, where cos has value 0. Hence $\frac{1}{\cos(z)}$ is analytic at all points on C and enclosed by C. Thus

$$\int_C \frac{1}{\cos(z)} dz = 0, \text{ by Cauchy-Goursat}$$

4. (25 points) Let the domain D be the complex plane minus the non-positive part of the real axis. Let $Log(z)$ be the principal branch of the logarithm function with argument in $(-\pi, \pi)$. Set $z^c := e^{cLog(z)}$, for every complex number c.

10

(a) Prove that if $c \neq -1$, then the branch $\frac{1}{c+1}e^{(c+1)Log(z)}$ of $\frac{1}{c+1}z^{c+1}$ is an anti-derivative of z^c in D.

Chain Rule

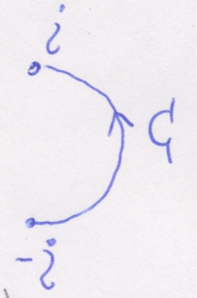
$$\frac{\partial}{\partial z}\left[\frac{1}{c+1} e^{(c+1)Log(z)}\right] = \frac{1}{\cancel{(c+1)}} \cancel{(c+1)} \frac{1}{z} e^{(c+1)Log(z)} = \frac{1}{e^{Log(z)}} e^{(c+1)Log(z)} =$$

$$= e^{c\,Log(z)}$$

$z = e^{Log(z)}$

15

(b) Let C be the semicircle from $-i$ to i parametrized by $z(\theta)=e^{i\theta}$, $-\pi/2\le\theta\le\pi/2$. Prove the equality $\int_C z^i dz = (e^{\pi/2}+e^{-\pi/2})(1+i)/2$.



C lies in the domain D.

Define $z^i = e^{i\,\mathrm{Log}(z)}$, so $c=i$ in Part (a).

Then $z^i = F'(z)$, where $F(z)=\frac{1}{i+1}e^{(i+1)\mathrm{Log}(z)}$, by Part a. So

$$\int_C z^i\,dz = F(i)-F(-i) =$$

$$= \frac{1}{i+1} e^{(i+1)\overbrace{\mathrm{Log}(i)}^{\pi i/2}} - \frac{1}{i+1} e^{(i+1)\overbrace{\mathrm{Log}(-i)}^{-\pi i/2}} =$$

$$= \frac{1}{1+i}\Big[\underbrace{e^{-\pi/2+\pi i/2}}_{ie^{-\pi/2}} - \underbrace{e^{\pi/2-\pi i/2}}_{-ie^{\pi/2}}\Big] = \underbrace{\frac{i}{1+i}}_{\frac{i(1-i)}{2}=\frac{1+i}{2}}\Big[e^{-\pi/2}+e^{\pi/2}\Big] =$$

$$= \frac{1+i}{2}\Big[e^{\pi/2} - e^{-\pi/2}\Big],$$

centered at 0

5. (25 points) Let C_R be the circle of radius R oriented counterclockwise, $R \geq 3$.

10

(a) Show that $\left|\int_{C_R} \frac{z^2+3}{z^4+6z^2+5}dz\right| \leq \frac{2\pi R(R^2+3)}{(R^2-1)(R^2-5)}$.

Hint: In order to bound the absolute value of a fraction from above one needs to bound the absolute value of the numerator from above and bound the absolute value of the denominator from *below*. Justify each of these bounds.

If $|z|=R$, then $|z^2+3| \leq |z^2|+3 = R^2+3$.

$$\underbrace{|z^4+6z^2+5|}_{(z^2+1)(z^2+5)} = \underbrace{|z^2+1|}_{\geq \, ||z^2|-1| \, = \, R^2-1} \underbrace{|z^2+5|}_{\geq \, ||z^2|-5| \, = \, R^2-5} \geq (R^2-1)(R^2-5).$$

(since $R^2 \geq 9$)

Hence, if $f(z) = \frac{z^2+3}{z^4+6z^2+5}$, then for $|z|=R$,

$$|f(z)| \leq \frac{R^2+3}{(R^2-1)(R^2-5)} \underset{\text{def}}{=:} M_R.$$

$$\left|\int_{C_R} f(z)\,dz\right| \leq M_R \underbrace{\text{length}(C_R)}_{2\pi R} = \frac{2\pi R(R^2+3)}{(R^2-1)(R^2-5)}$$

10

(b) Set $I_R := \int_{C_R} \frac{z^2+3}{z^4+6z^2+5} dz$. Prove the equality $I_R = I_3$, for all $R \geq 3$. Do this by stating the Principle of Deformation of the Path and showing that all its hypotheses are satisfied.

Principle of Deformation of the Path: Let C and C_1 be simple closed contour, with C_1 enclosed by C, oriented counterclockwise. Let f be analytic at all points on C and C_1 and in between. Then $\int_C f(z)dz = \int_{C_1} f(z)dz$.

The function $f(z) = \frac{z^2+3}{z^4+6z^2+5} = \frac{z^2+3}{(z^2+1)(z^2+5)}$ is analytic at all points of $\mathbb{C}$ except $\pm i$, $\pm\sqrt{5}\, i$. Hence, it is analytic at all points on C_R, C_3 and in between.

Thus $I_R = \int_{C_R} f(z)dz = \int_{C_3} f(z)dz = I_3$,

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(c) Prove that $I_3 = 0$ (and hence so is I_R, for $R \geq 3$). Hint: Compute $\lim_{R\to\infty} |I_R|$ in two ways, once using part 5a and once using part 5b.

$\lim_{R\to\infty} |I_R| = |I_3|$ by part (b)

$$0 \leq |I_R| \underset{\text{Part a}}{\leq} \frac{(2\pi R)\, R^2+3}{(R^2-1)(R^2-5)} = \frac{2\pi\left(\frac{1}{R^2}+\frac{3}{R^4}\right)}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{5}{R^2}\right)} \xrightarrow{R\to\infty} 0.$$

Hence, $|I_3| = \lim_{R\to\infty} |I_R| \underset{\substack{\uparrow \\ \text{Squeeze Thm}}}{=} 0.$ Thus $I_3 = 0$.