

8 pts

$$= \ln|x+iy|$$

1. (25 points) a) Prove that the function $u(x,y) = \frac{1}{2} \ln(x^2 + y^2)$ is harmonic on

Method 1: $\mathbb{R}^2 \setminus \{0\}$ (on the whole plane minus the origin).

u is the real part of the principal logarithm $\text{Log}(x+iy) = \ln|x+iy| + i \text{Arg}(x+iy)$, which is analytic on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, hence it is harmonic.

(The assumption that its first and second partials exist and are continuous follows from the Corollary in Sec 52, and can also be verified directly.) u is also the real part of the branch of \log with argument in $(0, 2\pi)$ analytic on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Hence it is harmonic on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$.

Method 2; $u_x = \frac{1}{2} \frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2}$, $u_y = \frac{y}{x^2+y^2}$

$$u_{xx} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} \quad \cdot \quad u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$u_{xx} + u_{yy} = 0 \quad \checkmark \quad \text{1-st \& 2-nd partials are continuous}$$

(as are u_{xy} and u_{yx}), on $\mathbb{R}^2 \setminus \{(0,0)\}$,

8 pts

b) Let v be a harmonic conjugate of the function u in part (a) in some open subset

D of $\mathbb{C} \setminus \{0\}$. Show that for all $z = x+iy$ in D , the equality $u_x(x,y) + iv_x(x,y) = \frac{1}{z}$

holds. (You do **not** need to find v).

$u+iv$ is analytic in D , so the Cauchy-Riemann Equation hold.

Hence $v_x = -u_y = \frac{-y}{x^2+y^2}$, so

$$u_x(x,y) + iv_x(x,y) = \frac{x-iy}{x^2+y^2} = \frac{x+iy}{|x+iy|^2} = \frac{1}{x+iy}$$

4 pts

c) Keep the notation of part b). Set $F(z) = u(x, y) + iv(x, y)$. Show that $F'(z) = \frac{1}{z}$.

F is analytic in D , by definition of harmonic conjugate, so F' exists and equals $F'(x+iy) = u_x(x, y) + i v_x(x, y) =$

\uparrow
C.R. Theorem (or by def of deriv limit)

$$= \frac{1}{z}$$

\uparrow Part b

5 pts

d) Let u be the function in part (a). Is there a harmonic conjugate v of u defined on the whole of $\mathbb{C} \setminus \{0\}$? Find such a function v or show that it does not exist.

No, there does not exist a harmonic conjugate v of u defined on $\mathbb{C} \setminus \{0\}$. The proof is by contradiction.

Assume there was. Then $F := u + iv$ is an antiderivative of $\frac{1}{z}$ on $\mathbb{C} \setminus \{0\}$, by part c. Hence,

$$\int_C \frac{1}{z} dz = 0 \text{ for every closed contour lying in } \mathbb{C} \setminus \{0\}.$$

But if C is the unit circle $\{z(\theta) = e^{i\theta}; 0 \leq \theta < 2\pi\}$,

$$\text{then } \int_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} i e^{i\theta} d\theta = i \int_0^{2\pi} d\theta = 2\pi i \neq 0.$$

A contradiction.

