1. (30 points) Compute the following (in cartesian or polar form):

8 pts

a) The polar form of $z := \dfrac{2}{-\sqrt{3}+\sqrt{3}i}$. $= \dfrac{2(-\sqrt{3}-\sqrt{3}i)}{(\sqrt{3})^2+(\sqrt{3})^2} = \dfrac{-1}{\sqrt{3}} \underbrace{(1+i)}_{\sqrt{2}e^{\pi i/4}} = -$

$= -\dfrac{\sqrt{2}}{\sqrt{3}} e^{\pi i/4} = \dfrac{\sqrt{2}}{\sqrt{3}} e^{5\pi i/4}$

4-pts

b) $|z^6|$, where $z$ is given in part a. $\left(\sqrt{\dfrac{2}{3}}\right)^6 = \left(\dfrac{2}{3}\right)^3 = \dfrac{8}{27}$

$(-\pi, \pi]$

6-pts

c) $\text{Log}(a^6)$, where $a = 2e^{-i[2\pi/11]}$. $a^6 = 2^6 e^{-i\,12\pi/11} = 2^6 e^{i(10\pi/11)}$

$\text{Log}(a^6) = \underbrace{\ln(2^6)}_{6\ln(2)} + i\,\dfrac{10\pi}{11}$

6-pts

d) Find all values of $(1-i)^{1/6}$. How many different values are there?

$$(1-i)^{1/6} = e^{1/6 \log(1-i)} \underset{1-i=\sqrt{2}\,e^{-\pi i/4}}{=\!=\!=} e^{1/6\,[\ln(\sqrt{2})+i(-\frac{\pi}{4}+2k\pi)]}, \quad k \text{ integer},$$

$$= \left(2^{1/2}\right)^{1/6} \cdot e^{-\frac{\pi i}{24}} \cdot e^{\frac{ik\pi}{3}}, \quad 0 \le k \le 5$$

$$= 2^{1/12}\, e^{-\pi i/24} \cdot e^{ik\pi/3}, \quad \text{''}$$

. There are 6 different values.

(2 pts)

6-pts

e) Find all values of $2^{3-i}$. How many different values are there? Justify!

$$2^{3-i} = e^{(3-i)\log(2)} = e^{(3-i)[\ln(2)+2k\pi i]}, \quad k \text{ integer},$$

$$= e^{(3\ln(2)+2k\pi)+i(-\ln(2)+6k\pi)}, \quad k \text{ integer},$$

$$= e^{[(3\ln(2)+2k\pi]} \cdot e^{-\ln(2)i}, \quad \text{''} \quad \text{''}.$$

There are $\infty$-many values

(2 pts)

$h(z) = $

2. (18 points) Find the set of points in the plane where the function $\dfrac{e^{\sin(z)}}{\sin(z)+\cos(z)}$ is analytic. Justify your answer.

Step 1: (9 pts)

The numerator is $f(g(z))$, where $f(z) = e^z$ and $g(z) = \sin(z)$. Both $f, g$ are entire (analytic everywhere in $\mathbb{C}$) and so $e^{\sin(z)}$ is analytic everywhere (by the Chain Rule). The denominator is a sum of entire functions, hence entire. Thus, by the Quotient Rule $h(z)$ is analytic at $z$ if and only if $\sin(z) + \cos(z) \neq 0$.

Step 2: (9 points) Solve for $\sin(z) + \cos(z) = 0$ (*) Multiply by $2i$

(+4) $\iff \dfrac{e^{iz} - e^{-iz}}{2i} = \dfrac{e^{iz} + e^{-iz}}{2}$

(*) becomes: $-\left(e^{iz} - e^{-iz}\right) = i\left(e^{iz} + e^{-iz}\right)$ Multiply by $e^{iz}$

$\iff -e^{2iz} + 1 = i\left(e^{2iz} + 1\right)$

$-i + 1 = e^{2iz}(i+1)$

$\iff e^{2iz} = \dfrac{1-i}{1+i} = \dfrac{\sqrt{2}e^{-\pi i/4}}{\sqrt{2}e^{+\pi i/4}} = e^{-\pi i/2}$

$\iff 2iz = -\pi i/2 + 2k\pi i$, $k$ integer

$\iff z = \dfrac{-\pi}{4} + k\pi$, $k$ integer.

Conclusion:

$h(z)$ is analytic at all points except

$\left\{ \dfrac{-\pi}{4} + k\pi,\ k \text{ integer} \right\}$.

6 pts

3. (18 points) a) Prove that the function $u(x, y) = e^{-y}\cos(x) + 3x^2y - y^3$ is harmonic on the whole of $\mathbb{R}^2$.

$u_x = -e^{-y}\sin(x) + 6xy$ $\qquad$ $u_y = -e^{-y}\cos(x) + 3x^2 - 3y^2$

$u_{xx} = -e^{-y}\cos(x) + 6y$ $\qquad$ $u_{yy} = e^{-y}\cos(x) - 6y.$

The Laplace Eq $u_{xx} + u_{yy} = 0$ is satisfied.

The partials above are continuous as are $u_{xy} = e^{-y}\sin(x) + 6x = u_{yx}$. Hence, $u$ is harmonic, by definition.

12 pts

b) Find a harmonic conjugate $v$ of the function $u$.

We are looking for $v$, such that $u + iv$ is analytic, so satisfies the Cauchy-Rieman Equations

(1) $u_x = v_y$ and (2) $u_y = -v_x$.

$$v = \int v_y\,dy \underset{(1)}{=} \int u_x\,dy = \int -e^{-y}\sin(x) + 6xy\,dy = e^{-y}\sin(x) + 3xy^2 + h(x)$$

where $h(x)$ satisfies

$$v_x = e^{-y}\cos(x) + 3y^2 + h'(x)$$

$$\overset{(2)}{=} -u_y = -(-e^{-y}\cos(x) + 3x^2 - 3y^2) = e^{-y}\cos(x) + 3y^2 - 3x^2$$

So, $h'(x) = -3x^2$. Hence $\boxed{h(x) = -x^3 + C.}$

$$\boxed{v(x,y) = e^{-y}\sin(x) + 3xy^2 - x^3 + C.}$$

16 pts

4. (18 points) (a) Let $f$ be a function analytic in the unit disk $D := \{z \;:\; |z| < 1\}$. Prove that if $\mathrm{Re}(f(z)) = 7$, for all $z$ in $D$, then $f(z)$ is constant throughout $D$. Provide the full and precise statements of the two theorems you use and explain
4 pt why their hypotheses are satisfied.

Thm 1: If $\underset{x+iy}{f(z)} = u(x,y) + iv(x,y)$ is analytic on an open set $D$,

then (1) $u_x = v_y$ and (2) $u_y = -v_x$ (the Cauchy Riemann equations)
are satisfied at all points of $D$.

4 pts
Thm 2: If $g$ is a real valued on a domain (open and path connected subset of $\mathbb{R}^2$) $D$ and $g_x$ and $g_y$ vanish in $D$, then $g$ is constant in $D$.

Answer:
Write $f(x+iy) = u(x,y) + iv(x,y)$.

We are given that $u(x,y) = \mathrm{Re}(f(x+iy)) = 7$. Hence $u_x = 0$ and $u_y = 0$.
By Theorem 1, we get that
$v_x \underset{(2)}{=} -u_y \equiv 0$ and $v_y \underset{(1)}{=} u_x \equiv 0$

Hence, the partials of $v$ vanish in $D$.
The unit disk $D$ is a domain.
Hence, $v$ is constant in $D$, by Theorem 2.
Thus both $u$ and $v$ are constant, and so
$f$ is constant in $D$.

Q.E.D

2 pts

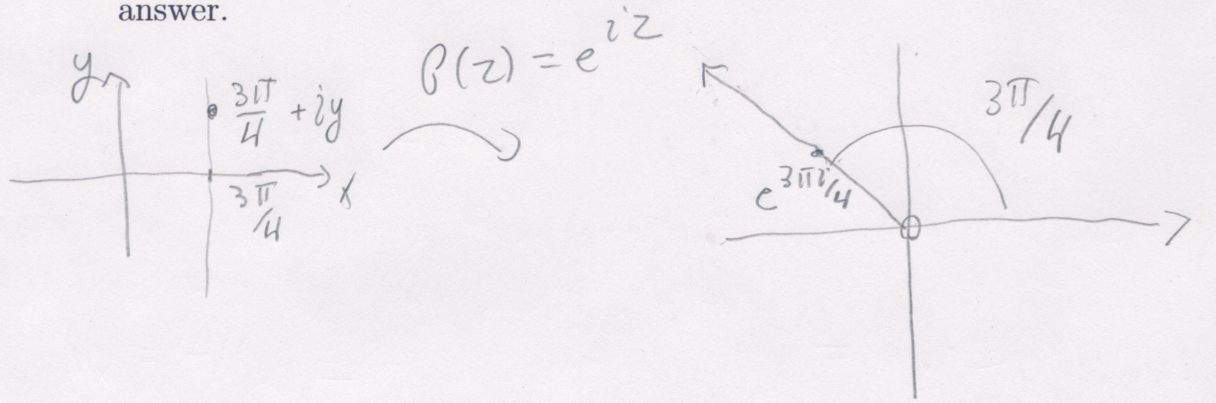
(b) Is the statement in part (a) true if $D$ is instead the complement of the $x$-axis $D = \{x + iy$ such that $x \neq 0\}$? Justify your answer.

No, the statement is false. As a counter example take $f(x+iy) = \begin{cases} 7, & \text{if } y>0 \\ 7+5i, & \text{if } y<0. \end{cases}$

The complement of the x-axis is not connected, hence not a domain.

9 pts

5. (18 points) a) Find the image of the vertical line $x = \frac{3\pi}{4}$ under the function $f(z) = e^{iz}$. Carefully draw the image and describe it in words. Justify your answer.

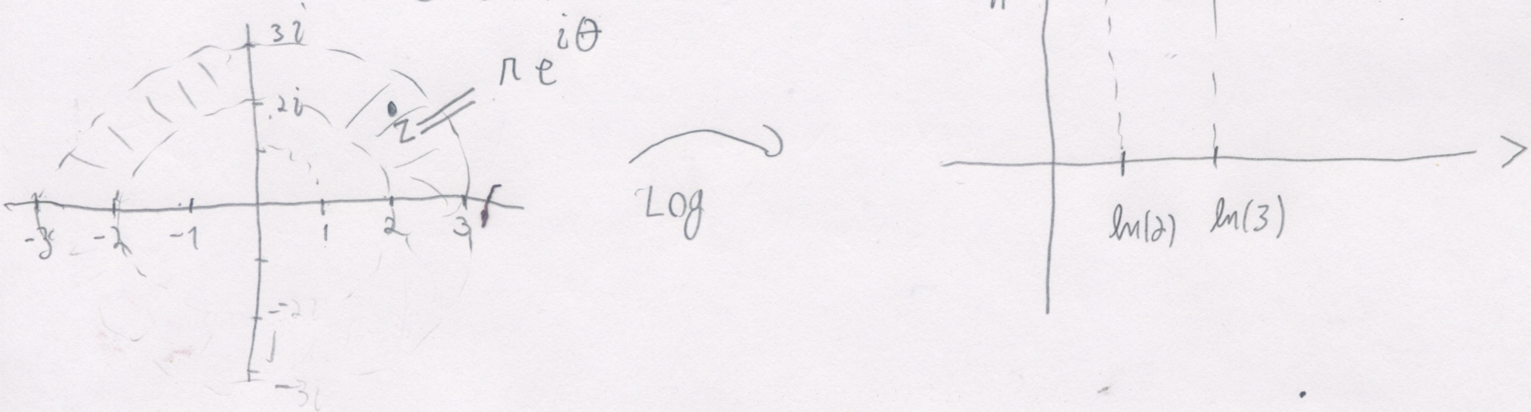



$$e^{i\left(\frac{3\pi}{4} + iy\right)} = e^{-y} \cdot e^{3\pi/4\, i}$$

The image of the vertical line $x = \frac{3\pi}{4}$ is the open ray emanating from the origin through $e^{3\pi i/4}$.

b) Let Log be the principal branch of the logarithm function with argument in the open interval $(-\pi, \pi)$. Describe geometrically the image of the set
$\{z$ such that $2 < |z| < 3$ and $\mathrm{Im}(z) > 0\}$
via the function Log. Carefully draw the image and describe it in words. Justify your answer.

$2 < r < 3, \quad 0 < \theta < \pi$




If $2 < r < 3$ and $0 < \theta < \pi$, then

$$\mathrm{Log}(re^{i\theta}) = \underbrace{\ln(r)}_{x} + i\underbrace{\theta}_{y}, \quad \text{where}$$

$$\ln(2) < x < \ln(3) \qquad 0 < y < \pi$$

The image is the open rectangle

$$\{(x+iy) : \ln(2) < x < \ln(3) \text{ and } 0 < y < \pi\}.$$