Review: There are three types of isolated singular points:

Pole: $\frac{e^z}{z^3} = \underbrace{\frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2}\frac{1}{z}}_{\text{Principal part}} + \frac{1}{3!} + \frac{z}{4!} + \cdots + \frac{z^n}{(n+3)!}$

Principal part has finitely many terms.

Behavior: $\lim_{z \to 0} |f(z)| = +\infty$.

Essential: $e^{1/z} = \cdots \frac{1}{n! z^n} + \cdots + \frac{1}{2z^2} + \frac{1}{z} + 1$

Principal part has ∞-many terms.

Removeable: $\frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots$

Principal part $= 0$.

Behavior: $\lim_{z \to 0} \frac{\sin(z)}{z} = 1$.

The function extends to an analytic function at 0.

Residues at poles;

Example: $\operatorname{Res}_{z=1+i} \dfrac{z^5+2z-3}{(z-1-i)^4} = f(z)$

Numerator $N(z) = z^5+2z-3$ has a Taylor series centered at $z_0 = 1+i$

$$N(z) = N(1+i) + \dots + \frac{N^{(3)}(1+i)}{3!}(z-1-i)^3$$

$$\frac{N(z)}{(z-1-i)^4} =$$ the coeff of $\dfrac{1}{z-1-i}$ will be

So $$\operatorname{Res}_{z=1+i} f(z) = \frac{N^{(3)}(1+i)}{3!} =$$

$$= \frac{5\cdot 4\cdot 3\cdot (1+i)^2}{3!} = 10\cdot(1+i)^2.$$

__Theorem:__ Let z_0 be an isolated singular point of a function f. Then

1) z_0 is a pole of order $m > 0$, if and only if f can be written in the form
$$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$
where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$.

2) Moreover, in that case,
$$\operatorname*{Res}_{z=z_0} f = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

__Note:__ If $m=1$ we get $\operatorname*{Res}_{z=0} f = \phi(z_0)$.

__Proof:__ Part 2 was explained.

__Part 1:__ Suppose that z_0 is a pole of order m. Then f has a Laurent series in a punctured disk centered at z_0

of the form

$$f(z) = \underbrace{\frac{\overset{\neq 0}{b_m}}{(z-z_0)^m} + \cdots + \frac{b_1}{(z-z_0)^1}}_{\text{principal part}} + a_0 + a_1(z-z_0) + \cdots$$

So $f(z) = \dfrac{(z-z_0)^m f(z)}{(z-z_0)^m}$ and

$$\phi(z) = (z-z_0)^m f(z) = \overset{\neq 0}{b_m} + b_{m-1}(z-z_0) + \cdots$$

has a removable singularity at z_0 so it extends to an analytic function at z_0

$\phi(z_0) = b_m \neq 0.$

Conversely, if $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$, $\phi(z_0) \neq 0$

and ϕ is analytic at z_0

then ... $\square$

Residue at a simple pole:

Example: (Use the following Lemma)

$$\operatorname*{Res}_{z=e^{\pi i/4}} \left(\frac{\overbrace{z^2}^{=f}}{\underbrace{z^4+1}_{=g}} \right)$$

Lemma: Assume that f, g are analytic in some open set containing z_0 and

1) $f(z_0) \neq 0$
2) $g(z_0) = 0$
3) $g'(z_0) \neq 0$

Then f/g has a pole of order 1 at z_0 and

$$\boxed{\operatorname*{Res}_{z=0} \left(f/g \right) = \frac{f(z_0)}{g'(z_0)}.}$$

Back to Example: conditions 1,2,3 hold. So

$$\operatorname*{Res}_{z=e^{\pi i/4}} \left(\frac{z^2}{z^4+1} \right) = \frac{\left(e^{\pi i/4}\right)^2}{4\left(e^{\pi i/4}\right)^3} =$$

$$= \frac{1}{4} e^{-\pi i/4} = \frac{1}{4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right).$$

Proof of the Lemma:

The Taylor series of g centered at z_0 is

$$g'(z_0)(z-z_0) + \cdots \text{ positive powers } (z-z_0)$$

$$= (z-z_0)\cdot\Big[\underbrace{\underbrace{g'(z_0)}_{\neq 0} + \text{sum of positive powers of } z-z_0}_{\phi(z)}\Big]$$

So
$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-z_0)\,\phi(z)} = \frac{f(z)/\phi(z)}{(z-z_0)}.$$

So, by the previous Lemma

$$\operatorname*{Res}_{z=z_0} \frac{f(z)}{g(z)} = f(z_0)/\phi(z_0) = \frac{f(z_0)}{g'(z_0)}. \quad \square$$

Behavior of f near an isolated singular point z_0:

a) Removable singularity.

$$\lim_{z\to z_0} f(z) \text{ exists.}$$

b) <u>Pole</u>: If z_0 is a pole of f.

In that case $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$,

for some $m > 0$, where ϕ is analytic and $\phi(z_0) \neq 0$. Then

$z \to z_0$ $\to |\phi(z_0)| \neq 0$

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \frac{|\phi(z)|}{|z-z_0|^m} = +\infty$$

$z \to z_0$ $\to 0$

c) <u>Essential</u>: If z_0 is an essential singularity of f we have:

<u>Theorem</u> (Cassorati-Weierstrass): Let z_0 be an essential singularity of a function f.

Then for every complex number w there exists a sequence $\{z_n\}_{n=1}^{\infty}$ such that

1) $\lim\limits_{n \to \infty} z_n = z_0$

2) $\lim\limits_{n \to \infty} f(z_n) = w$.

The Argument Principle

<u>Def</u>: Let U be an open subset of the complex plane. A function f is said to be MEROMORPHIC, if f is analytic throughout U, except for poles.

<u>Lemma</u>: Assume that f is a non-constant and meromorphic function on an open set U and z_0 is a point of U. Then

(1) There exists an integer m, such that

$$f(z) = (z-z_0)^m g(z),$$

where g is analytic at z_0 and $g(z_0) \neq 0$.

- If $m > 0$, then m is called the MULTIPLICITY of z_0 as a zero of f.
- If $m < 0$, then $|m|$ is called the MULTIPLICITY of z_0 as a pole of f.

(2) $\operatorname{Res}_{z=z_0}\left(\frac{f'}{f}\right) = m.$

Proof: (1) If $m>0$, we have seen already.

If $m<0$, then take $g(z) = (z-z_0)^{|m|}\cdot f(z)$

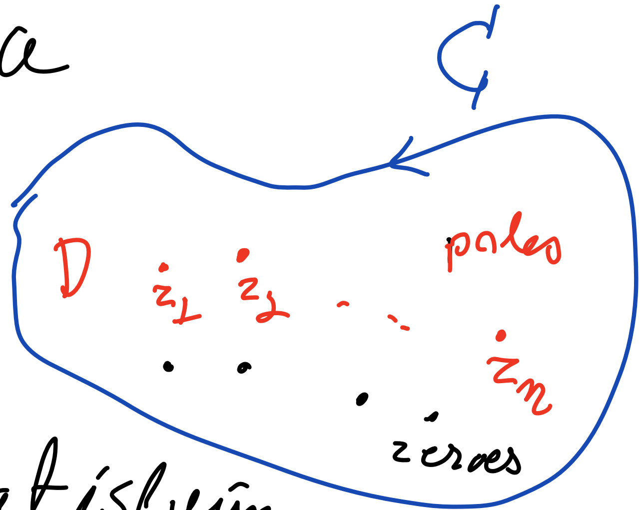
(2)

$$\operatorname{Res}_{z=z_0}\left(\frac{f'}{f}\right) = \operatorname{Res}_{z=z_0} \frac{m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z)}{(z-z_0)^m g(z)}$$

$$= \underbrace{\operatorname{Res}_{z=z_0} \frac{m}{z-z_0}}_{m} + \underbrace{\operatorname{Res}_{z=z_0}\left(\underbrace{\frac{g'(z)}{g(z)}}_{\text{analytic at } z_0}\right)}_{0} = m \quad \square$$

Theorem: (The Argument Principle)

Let C_1 denote a positively oriented simple closed contour and let f be a function satisfying



a) f is meromorphic in the domain D bounded by C_1.

b) f is analytic and non-zero at every point of C_1.

Then

$$\frac{1}{2\pi i}\int_{C_1} \frac{f'(z)}{f(z)}\,dz = \left(\begin{array}{c}\#\text{ of zeros of } f \text{ in D counted}\\ \text{with multiplicities}\end{array}\right) - \left(\begin{array}{c}\#\text{ of poles of}\\ f \text{ in D}\\ \text{counted with}\\ \text{multiplicities}\end{array}\right)$$

$\uparrow$ numbers

Proof: The union $D \cup C_1$ is a closed and bounded set. It follows that

f has only finitely many zeroes and finitely many poles in $D \cup C$. This follows from the Bolzano–Weierstrass Theorem.

$$\frac{1}{2\pi i}\int_{C}\frac{f'(z)}{f(z)}\,dz \overset{\text{Cauchy's Residue Thm}}{=}$$

$$= \sum_{z \text{ is an isolated sing}} \operatorname{Res}_z\left(\frac{f'}{f}\right) =$$

$$= \sum_{\substack{z \text{ is a zero} \\ \text{of } f}} \underbrace{\operatorname{Res}_z\left(\frac{f'}{f}\right)}_{\text{mult of } z \text{ as a zero of } f} + \sum_{\substack{z \text{ is a pole} \\ \text{of } f}} \underbrace{\operatorname{Res}_z\left(f'/f\right)}_{\text{minus the mult of } z \text{ as a pole}}$$

$$= \left(\begin{array}{c}\#\text{ of zeroes of } f \\ \text{in } D \text{ counted with} \\ \text{multiplicities}\end{array}\right) - \left(\begin{array}{c}\#\text{ of poles of } f \\ \text{in } D \text{ counted} \\ \text{with mult}\end{array}\right).$$

$\square$

Geometric Meaning of the Argument Principle:

$\frac{1}{2\pi i}\Delta \arg f(z)$ $\quad w(t) = f(z(t)) = |w(t)|\left(\cos(\theta(t)) + i\sin(\theta(t))\right)$

θ

$$\frac{1}{2\pi i}\int_{f(C)} \frac{1}{w}\,dw =$$ number of times the contour $f(C)$ winds about the origin in the positive direction.

$$= \frac{1}{2\pi i}\int_0^1 \frac{1}{f(z(t))} f'(z(t))\cdot z'(t)\,dt =$$

$$= \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \text{(\# zeroes of } f \text{ in } D \text{ counted with mult)} - \text{(\# of poles of } f \text{ in } D \ldots)$$