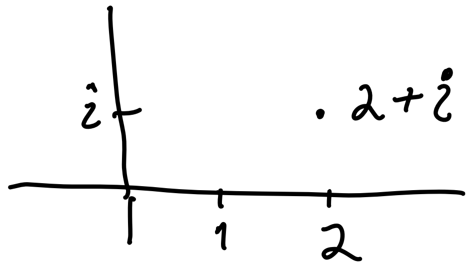
## Review:

Complex numbers are vectors in $\mathbb{R}^2$.
We write $x + iy$ instead of $(x, y)$.




**Addition:** Like vectors in $\mathbb{R}^2$

**Multiplication:** $i^2 = -1$, so

$$\boxed{(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)}$$

**Multiplicative Inverse:** If $z = x + iy \neq 0$, then

$$z^{-1} = \frac{x}{x^2+y^2} - \left(\frac{y}{x^2+y^2}\right)i$$

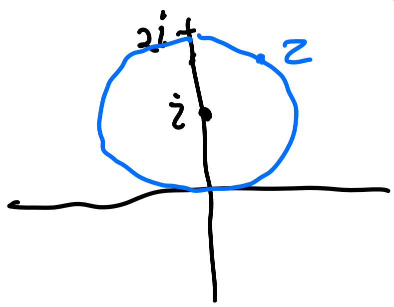
**Division:** $z_1/z_2 \overset{\text{definition}}{:=} z_1 \cdot z_2^{-1}$.

**Example:** $\frac{1+2i}{1+i} = (1+2i)\underbrace{(1+i)^{-1}}_{\frac{1-i}{2}} = \frac{3}{2} + \frac{1}{2}i$

**Absolute Value:** $|x+iy| = \sqrt{x^2+y^2}$.

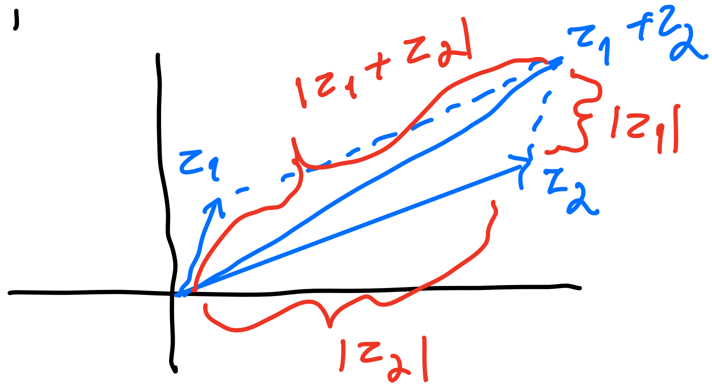
Distance between $z_1$ and $z_2$: $|z_1 - z_2|$.

**Example:** $\{z : |z - i| = 1\}$ is



## Triangle Inequality:

$|z_1 + z_2| \leq |z_1| + |z_2|$




Corollary: $|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$

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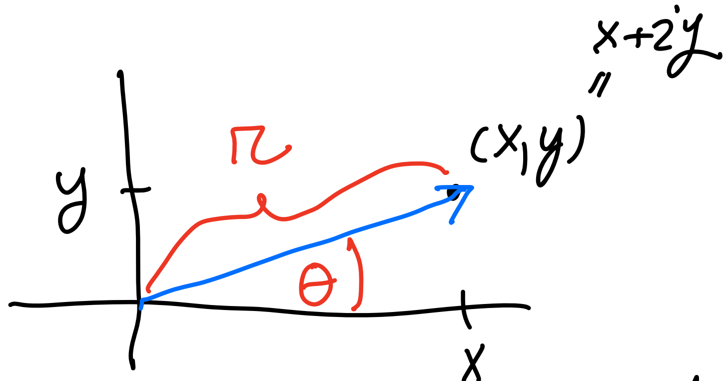
## Polar Coordinates:

Recall: A point in $\mathbb{R}^2$ with Cartesian coordinates $(x, y)$ has polar coordinates $(r, \theta)$, where

$x = r\cos(\theta)$
$y = r\sin(\theta)$




Note: If $z = x+iy$, then $r$ in its polar coordinates is just $|z|$.

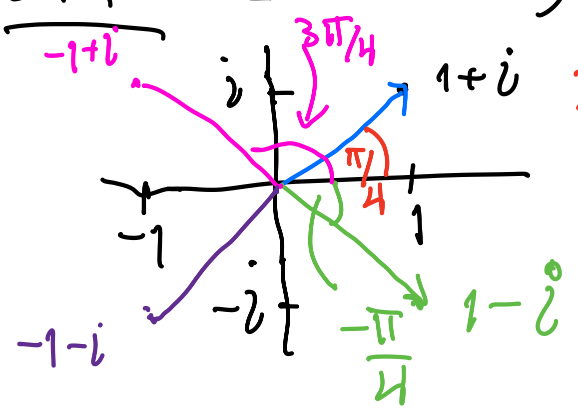
We call $\theta$ an argument of the complex number $z$. If $\theta$ is an argument for $z$, then so is $\theta + 2\pi$.

Def: 1) The argument, $\arg(z)$, of a non-zero complex number $z = x+iy$, is the set of all $\theta$ such that

$$\begin{cases} x = |z|\cos(\theta), \text{ and} \\ y = |z|\sin(\theta) \end{cases}$$

2) The Principal Argument $\mathrm{Arg}(z)$ of a non-zero complex number $z$ is the unique argument in the interval $(-\pi, \pi]$.

Ex: $z = 1+i$, $\quad |z| = \sqrt{1^2+1^2} = \sqrt{2}$




$\frac{\pi}{4}$

$$\mathrm{Arg}(1+i) = \frac{\pi}{4}$$

$$\arg(1+i) = \frac{\pi}{4} + 2k\pi, \text{ k is an integer}$$

$$\mathrm{Arg}(1-i) = -\frac{\pi}{4}$$

$$\arg(1-i) = -\frac{\pi}{4} + 2k\pi, \text{ k is} \ldots$$

Ex: $-\frac{\pi}{4} + \underbrace{2\pi}_{\frac{8\pi}{4}} = \frac{7\pi}{4}$ is also an argument of $1-i$

$$\mathrm{Arg}(-1+i) = \frac{3\pi}{4}$$

$\operatorname{Arg}(-1-i) = -\frac{3\pi}{4}$

**Def:** (Euler's "Formula" Notation)

$$e^{i\theta} := \cos(\theta) + i\sin(\theta),$$

for any real numbers $\theta$

**Notation:** Using Euler's notation, we can write a complex number $z$ with polar coor $(|z|, \theta)$ in the form

$$z = |z| e^{i\theta} = \overset{r}{|z|}\left(\cos(\theta) + i\sin(\theta)\right)$$

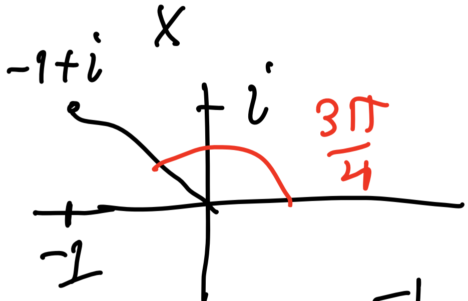
EXPONENTIAL FORM

$$= \underbrace{r\cos(\theta)}_{x} + i\,\underbrace{r\sin(\theta)}_{y}$$

**Ex:** $z = -1 + i$

$|z| = \sqrt{2}$

$$\boxed{z = \sqrt{2}\, e^{\frac{3\pi i}{4}}}$$




The complex plane

**Ex:** Let $\gamma : [0, \pi] \longrightarrow \mathbb{C} = \mathbb{R}^2$ be defined by

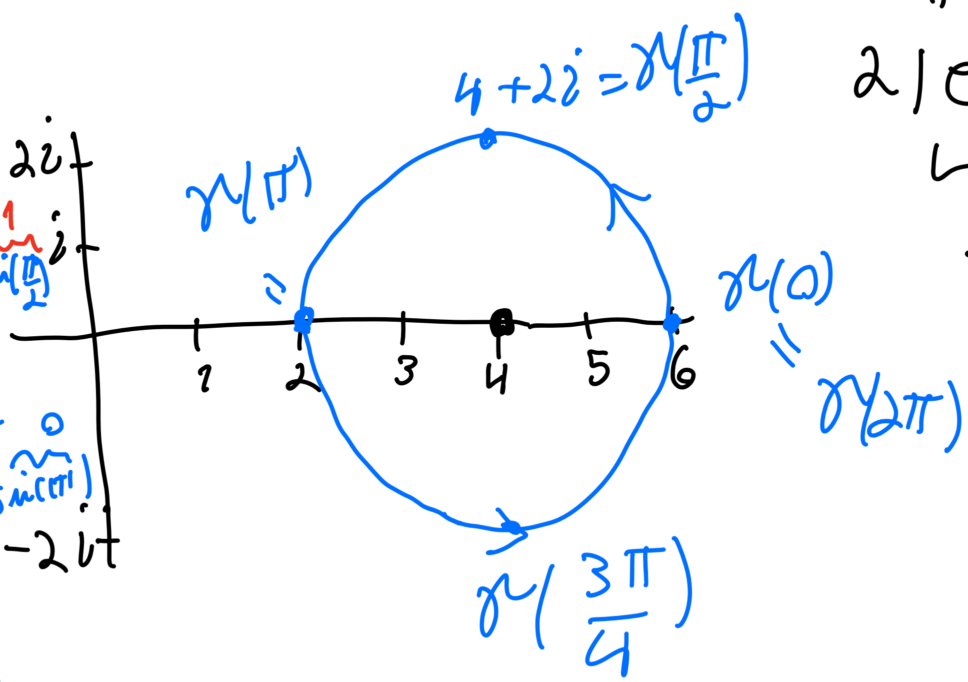
$$\boxed{\gamma(\theta) = 4 + 2e^{i\theta}}$$

Describe the image of $\gamma$

__Answer:__ Note that $|\gamma(\theta)-4| = |2e^{i\theta}| = 2$

$2\underbrace{|e^{i\theta}|}_{1}$

| $\theta$ | $\gamma(\theta)$ |
|---|---|
| $0$ | $6$ |
| $\frac{\pi}{2}$ | $4+2(\underbrace{\cos(\frac{\pi}{2})}_{0}+i\underbrace{\sin(\frac{\pi}{2})}_{1})$ $= 4+2i$ |
| $\pi$ | $4+2e^{\pi i} = 4+2(\underbrace{\cos(\pi)}_{-1}+i\underbrace{\sin(\pi)}_{0}) = 4-2=2$ |

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## Product and Quotients in exponential Form:

Recall the following two trig identities:

$$\boxed{\begin{aligned}\cos(\theta_1+\theta_2) &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\\ \sin(\theta_1+\theta_2) &= \cos(\theta_2)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1)\end{aligned}}$$

Let us evaluate

$$e^{i\theta_1} \cdot e^{i\theta_2} = (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2))$$

$$= \underbrace{[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)]}_{\cos(\theta_1+\theta_2)} + i\,\underbrace{[\cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1)]}_{\sin(\theta_1+\theta_2)}$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}.$$

$$\boxed{e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}}$$

<u>Conclusion!</u> The usual exponential identities hold

(1) $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

(2) $e^{i\theta_1} / e^{i\theta_2} = e^{i(\theta_1-\theta_2)}$

<u>Reason!</u> (1) was shown above,

(2) Multiply both sides by $e^{i\theta_2}$

$$LHS = e^{i\theta_1} = \underbrace{e^{i(\theta_1-\theta_2)}\, e^{i\theta_2}}_{\boxed{\text{by (1)}}\ \text{“}\ e^{i\theta_1}}$$

$\square$

**Cor:** If $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$,

then
(1)
$$\boxed{z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}}$$

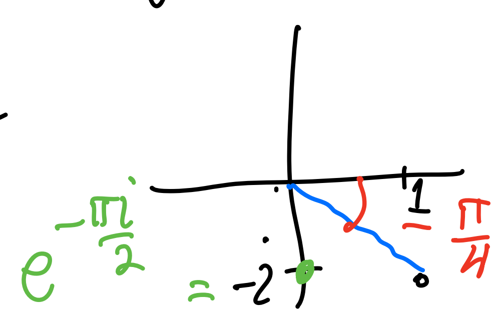
(2) If $z_2 \neq 0$, then
$$z_1/z_2 = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

(3) If $z = re^{i\theta}$, then for any integer $n$, we have
$$\boxed{z^n = r^n e^{i(n\theta)}}$$
$$z^n = re^{i\theta}\, re^{i\theta} \cdots re^{i\theta}$$

(4) (special case of 3)
$$\boxed{\left(\cos(\theta) + i\sin(\theta)\right)^n = \cos(n\theta) + i\sin(n\theta)}$$

**Example:** Evaluate $(1-i)^{10}$

**step 1:** convert $1-i$ to exponential form:
$$1-i = \underbrace{|1-i|}_{\sqrt{2}}\, e^{-\frac{\pi}{4}i} = \boxed{\sqrt{2}\, e^{-\frac{\pi}{4}i}}$$



$$(1-i)^{10} = \underbrace{(\sqrt{2})^{10}}_{(2^{1/2})^{10}} \cdot e^{-\frac{10\pi}{4}i} = 2^5 \cdot \underbrace{e^{-\frac{2\pi i}{4}}}_{\underbrace{e^{-\frac{\pi i}{2}}}_{-i}} = \underbrace{-2^5 i}_{-32i}$$

# Algebraic properties of $\underline{\arg(z)}$:

We had the identity

$$\underbrace{(r_1 e^{i\theta_1})}_{z_1} \underbrace{(r_2 e^{i\theta_2})}_{z_2} = \underbrace{r_1 r_2 e^{i(\theta_1+\theta_2)}}_{z_1 z_2}.$$

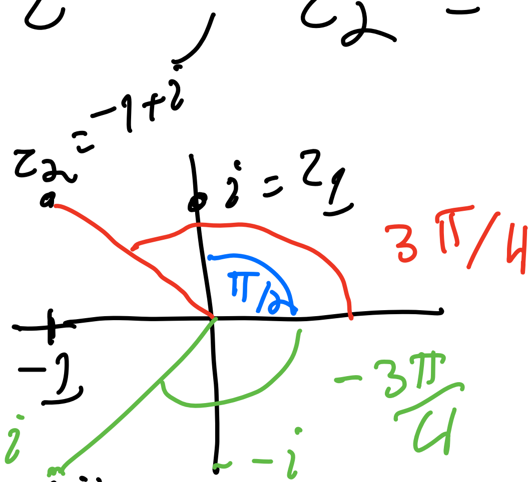
It follows that

$$\boxed{\underbrace{\arg(z_1 z_2)}_{\substack{\text{"} \\ \text{set of all} \\ \text{arguments of } z_1z_2}} \underset{(**)}{=} \underbrace{\arg(z_1)}_{\substack{\text{"} \\ \text{set of} \\ \text{all arguments} \\ \text{of } z_1}} + \underbrace{\arg(z_2)}_{\substack{\text{"} \\ \text{set of all} \\ \text{arguments} \\ \text{of } z_2}}.}$$

set of all sums of one argument of $z_1$ + one argument of $z_2$

The equality (**) does NOT hold for principal arguments.

<u>Ex!</u> Let $z_1 = i$, $z_2 = -1+i$




$$\operatorname{Arg}(z_1) = \operatorname{Arg}(i) = \frac{\pi}{2}$$

$$\operatorname{Arg}(z_2) = \operatorname{Arg}(-1+i) = \frac{3\pi}{4}$$

$$z_1 z_2 = i(-1+i) = -1-i$$

$$\operatorname{Arg}(z_1 z_2) = -\frac{3\pi}{4} \neq \underbrace{\underbrace{\operatorname{Arg}(z_1)}_{\frac{\pi}{2}} + \underbrace{\operatorname{Arg}(z_2)}_{\frac{3\pi}{4}}}_{\frac{5\pi}{4}}$$

$$\| $$

$$-2\pi + \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2$$

# Roots of complex numbers.

(3) If $z = re^{i\theta}$, then for any integer $n$, we have

$$\boxed{z^n = r^n e^{i(n\theta)}}$$

<u>Example:</u> Find all 4-th roots of 1, so all solution to

$$\boxed{z^4 = 1}.$$

Write $z = \hat{r} e^{i\theta}$ so

$$z^4 \underset{(3)}{=} r^4 e^{i\boxed{4\theta}} = 1 = 1e^{\boxed{0}i}$$

$\boxed{r = 1}$. $4\theta = 2k\pi$, $k$ is an integer.

$\theta = \frac{2k\pi}{4} = k\frac{\pi}{2}$, $k$ integer

a repetition

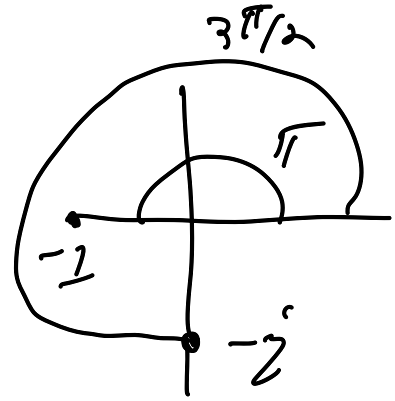
$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \cancel{2\pi}$$

$$k = 0 \quad 1 \quad 2 \quad 3 \quad \cancel{4}$$

Can take $0 \le k \le 3$

$$z = 1\cdot e^{0i}, \quad e^{\frac{\pi}{2}i}, \quad e^{\pi i}, \quad e^{\frac{3\pi i}{2}}$$

$$\quad\;\; = 1 \qquad\;\; = i \qquad = -1 \qquad = -i$$




There are precisely 4 fourth roots of 1.

More generally, the set of $n$-th roots of 1, i.e., the set of solutions of

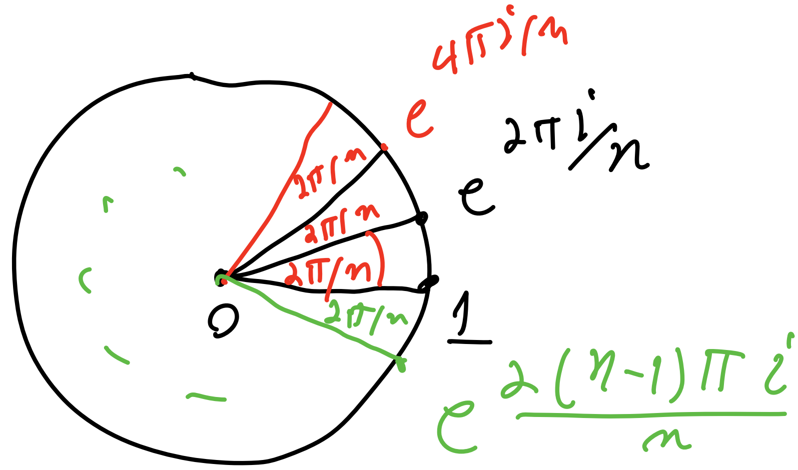
$$z^n = 1$$

$z = e^{i\theta}$, where $n\theta = 2k\pi$, $k$ integer

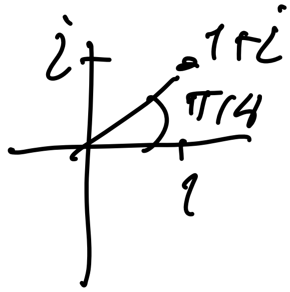
$$\theta = \frac{2k\pi}{n}, \quad 0 \le k \le n-1$$

There are precisely $n$ $n$-th roots of 1.

$$z = 1, \; e^{\frac{2\pi i}{n}}, \; e^{\frac{4\pi i}{n}}, \; \dots, \; e^{\frac{2k\pi}{n}}, \; \dots, \; e^{\frac{2(n-1)\pi i}{n}}$$




Examples: Find all cube roots of $z = \underbrace{1+i}_{\sqrt{2}\, e^{\pi i/4}}$.




We need to find all solution $w$ of $w^3 = 1+i = \sqrt{2}\, e^{\pi i/4}$

Write $w = r e^{i\theta}$

$w^3 = r^3 e^{i(3\theta)}$

So $r^3 = \sqrt{2} = 2^{1/2}$, so $r = \left(2^{1/2}\right)^{1/3} = 2^{1/6}$.

$3\theta = \frac{\pi i}{4} + 2k\pi$

$\theta = \frac{\pi i}{12} + \frac{2\pi}{3}\cdot k$, $k$ integers

$0 \le k \le 2$

$w = 2^{1/6} e^{\pi i/12}, \; 2^{1/6} e^{\left[\frac{\pi}{12} + \frac{2\pi}{3}\right] i}, \; 2^{1/6} e^{\left[\frac{\pi}{12} + \frac{4\pi}{3}\right] i}$

Precisely three solution.

<u>Observation</u>: $z = 1+i = \sqrt{2}\, e^{\pi i/4}$.

To find all cube roots of $z$ we follow the following two steps.

<u>Step 1</u>: Find one sol'n: $w_0 = r e^{i\theta_0}$,

$w_0^3 = \underbrace{r^3}_{\substack{\| \\ \sqrt{2} \\ \| \\ 2^{1/2}}} e^{i3\theta_0}$

$r = 2^{1/6}$

$3\theta_0 = \pi i/4$

$\theta_0 = \pi i/12$

Choose

One sol'n is $\boxed{w_0 = 2^{1/6} \cdot e^{\pi i/12}}$.

<u>Step 2</u>: Observe that if $\xi$ is a cube root of $1$ (i.e. $\xi^3 = 1$) then

$(w_0 \xi)^3 = \underbrace{w_0^3}_{z} \cdot \xi^3 = z \cdot 1 = z = 1+i$

So we find 3 - sol'n

$w_0 \cdot 1$, $w_0 \cdot \underbrace{e^{\frac{2\pi i}{3}}}_{= \xi}$, $w_0 \cdot \underbrace{e^{\frac{4\pi i}{3}}}_{\xi^2}$

<u>Note</u>: We have found all solutions, because if

$$w^3 = 1+i = w_0^3$$

Then $\left(\frac{w}{w_0}\right)^3 = \frac{w^3}{w_0^3} = \frac{1+i}{1+i} = 1$

So $w = w_0$ times a cube root of 1.

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<u>Conclusion</u>

If $z$ is a non-zero complex number, then it has $n$ $n$-th roots. The equation

$$w^n = z$$

has precisely $n$-solution

If $w_0$ is one solution, then the general sol'n is

$w_0 \cdot \xi$, where $\xi$ is an $n$-th root of 1