Math 421 sec 1 (32001) - Complex Variables - Fall 2024
TuTh 11:30 → 12:45 LGRC A201

Professor: Eyal Markman
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Office hours: Tuesday 1:00 → 2:00 pm, Thursday, 2:00 → 3:00 pm, and by appointment. Office hours are held in 1223G LGRT.
Course Web page: `http://people.math.umass.edu/~markman/` **Please check it often!**

Text: *Complex Variables and Applications,* 8-th Edition, by James Ward Brown and Ruel V. Churchill, McGraw-Hill.

Prerequisites: Math 233.

Homework: Will be assigned weekly and will be due each Thursday unless mentioned otherwise. The homework will be graded by a special grader. Due to lack of funds it will not be possible to grade all the homework problems assigned. A few of the homework problems will be corrected and graded every week. Nevertheless, for your own benefit, you will be asked to hand in *all* the homework problems assigned. Your grade on each homework assignment will be calculated as follows:

- 70% The grade on the corrected problems.
- 30% Credit for handing in *most* of the homework problems assigned. Partial credit will be given.

Late homework will not be collected. Instead, your three lowest grades will be dropped.
Grades:

Homework–20%
Two Midterms–50% (each 25%)
Final Exam –30%

First Midterm: Thursday, October 10, during class period.

Second Midterm: Thursday, November 14, during class period.

Final: To be scheduled by the registrar. Make-ups will not be given to accommodate travel plans.

Calculators Policy: Calculators will **not** be allowed in the exams. Calculators and computers may be used to check answers on the homework assignments. Nevertheless, an unsubstantiated answer will not receive credit.

See back . . .

Homework Assignment 1 (Due Thursday, September 12)

Section 2 page 5: 4

Section 3 page 8: 1 (a), (b)

Section 4 page 12: 4, 5 (a), (c), 6

Section 5 page 14: 1 (c), (d), 9, and the extra problem:

Use established properties of moduli to show that when $|z_3| \neq |z_4|$, then

$$\left|\frac{z_1 + z_2}{z_3 + z_4}\right| \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}$$

Section 8 page 22: 1 (a), 2, 3, 4, 5 (c), 5 (a), 6, 9, 10.

Check our website for likely additional problems.

Syllabus:

1) Complex Numbers: algebraic and geometric properties, polar form, powers and roots.

2) Analytic functions: Differentiability and Cauchy-Riemann equations, Harmonic functions, examples.

3) Elementary functions of a complex variable: exponential and trigonometric functions, logarithms.

4) Path integrals: contour integration and Cauchy's integral formula; Liouville's theorem, Maximum modulus theorem, the Fundamental Theorem of Algebra.

5) Series: Taylor and Laurant expansions, convergence, term-by-term operations with infinite series.

6) Isolated singularities and residues. Essential singularities and poles.

7) Evaluation of Improper integrals via residues.

If time permits:

8) Mappings by elementary functions and linear fractional transformations; conformal mappings.

Complex numbers (x, y) are just vectors in $\mathbb{R}^2$.

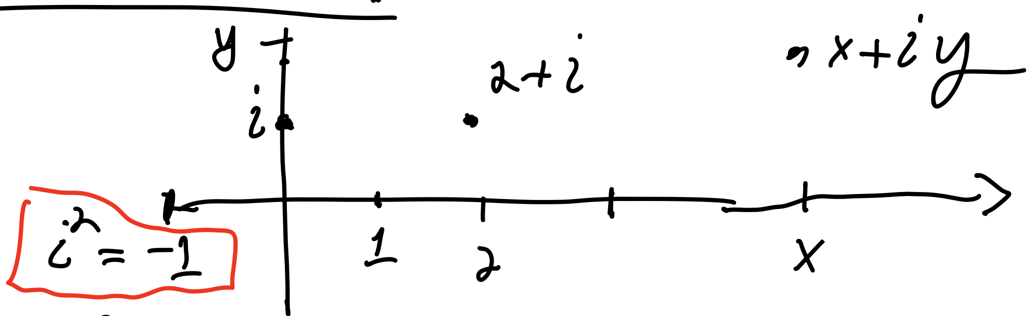
We usually write $x + iy$

<u>Addition:</u> Like vectors in $\mathbb{R}^2$

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

<u>Multiplication:</u>

$x + iy$



$$i^2 = (0, 1)^2 = -1$$

$$\boxed{(x_1 + i y_1)(x_2 + i y_2) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2) i}$$

$z = x + iy$, x, y real

x: Re, Real Part

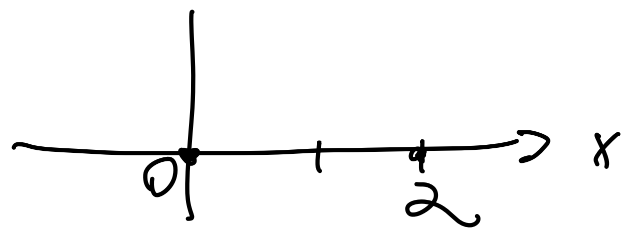
y: Im, Imaginary Part

$\mathrm{Re}(2+3i) = 2$, $\mathrm{Im}(2+3i) = 3$,

$$\boxed{(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, \; x_1 y_2 + y_1 x_2)}$$

The real numbers are identified with the points in the complex plane on the x - axis.

$$2 = 2 + 0 \cdot i$$



Algebraic Properties:

Addition, Multiplication,

- Zero complex number $0 + i0 = 0$

  $z + 0 = z$

- $1 \cdot z = z$

- Negative $\quad z = (x, y) = x + iy$

  $$-z = -x - y = -x - iy$$

- **Multiplicative Inverse:**

  If $z = x + iy \neq 0 + 0i$, then there exists a complex number z^{-1} such that $z \cdot z^{-1} = \underline{1}$

Example: $z = 1 + 2i$

$z^{-1} = u + iv$. Find u, v

$$1 + 0i = (1 + 2i)(u + iv) = (u - 2v) + i(2u + v)$$

$$u = 1 + 2\left(-\frac{2}{5}\right) = \frac{1}{5}$$

$$\left.\begin{aligned} u - 2v &= 1 \\ 2u + v &= 0 \end{aligned}\right\} \iff \begin{aligned} u - 2v &= 1 \\ 0 \cdot u + 5v &= -2 \end{aligned} \quad \text{so } v = -\frac{2}{5}$$

$$\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

↑ Add -2 Eq1 to Eq2

$$z^{-1} = u + iv = \frac{1}{5} - \frac{2}{5} i$$

In general: If $z = x + yi \neq 0$ then $z^{-1} = u + iv$, where (u, v) is the solution to

$$1 + 0i = (x + yi)(u + iv) = (xu - yv) + i(yu + xv)$$

$$\left.\begin{aligned} xu - yv &= 1 \\ yu + xv &= 0 \end{aligned}\right\} \quad \begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{x^2 + y^2}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{x^2+y^2} \begin{pmatrix} x \\ -y \end{pmatrix}$$

$$\boxed{z^{-1} = u + iv = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)} = \frac{x-iy}{x^2+y^2}$$

Q.E.D

<u>Division</u>: If z_1, z_2 are complex numbers, with $z_2 \neq 0$, then

$$z_1/z_2 \overset{def}{:=} z_1 \cdot (z_2^{-1})$$

<u>Ex</u>: $$\frac{1+2i}{1+i} = (1+2i)\underbrace{(1+i)^{-1}}_{\frac{1-i}{2}} = \frac{1}{2}\underbrace{(1+2i)(1-i)}_{3+i} = \frac{3}{2} + \frac{1}{2}i$$

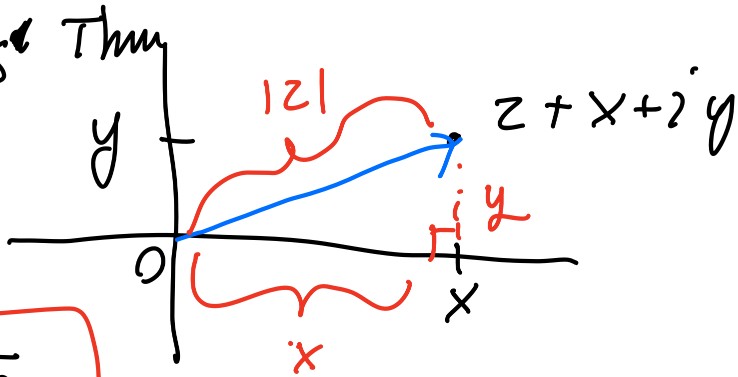
Absolute Value; (Modulus)

Let $z = x + iy$ be a complex number

Def 1; The absolute value $|z|$ is the distance of z from 0

Pythagoras' Thm

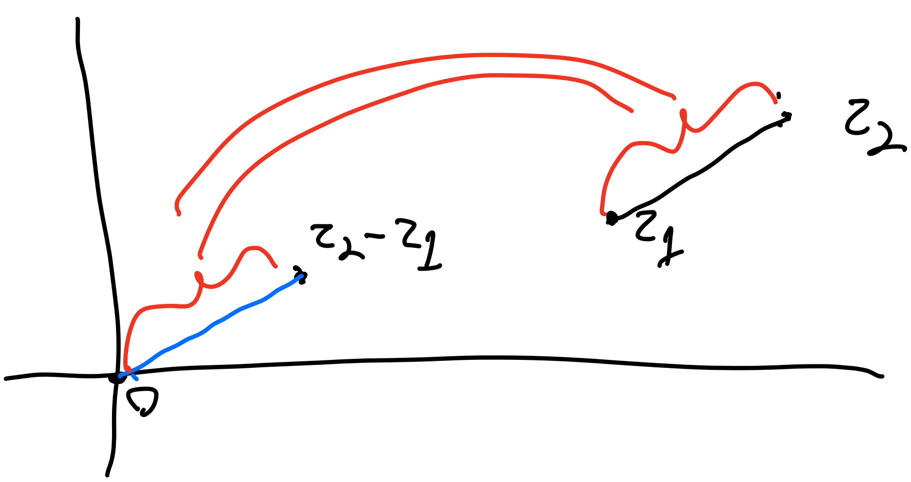
$$|z|^2 = x^2 + y^2$$



$$\boxed{|z| \overset{def}{:=} \sqrt{x^2+y^2}}$$

Ex! $|2+3i| = \sqrt{2^2+3^2} = \sqrt{13}$.

Def 2; The Distance between two complex numbers z_1 and z_2 is $|z_2 - z_1|$.

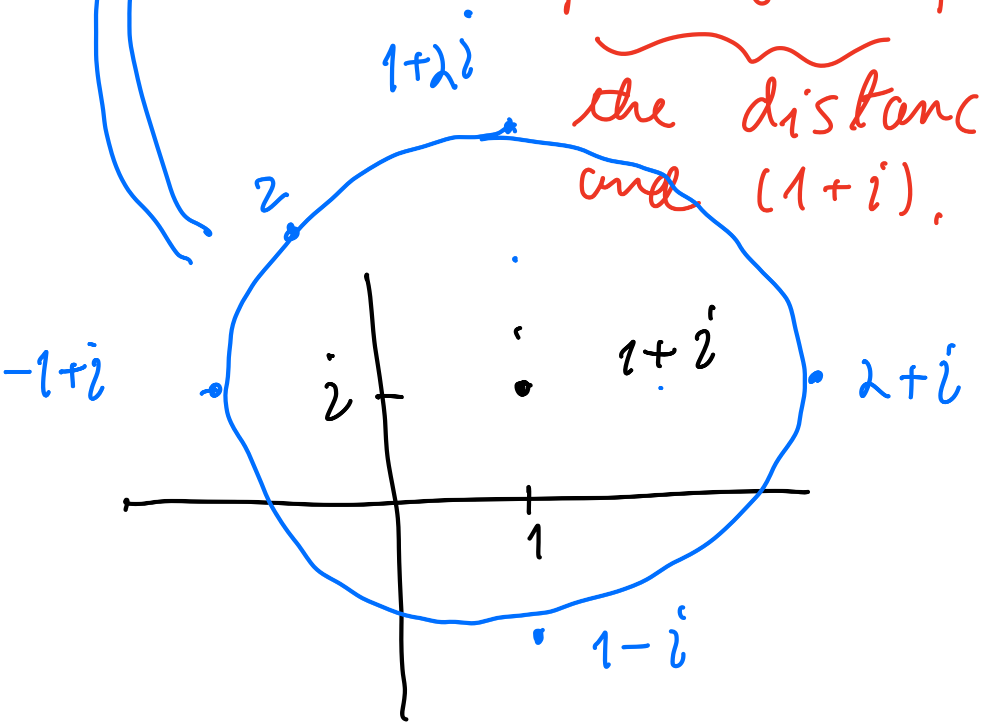


Ex: The distance between $2+3i$ and $1+i$ is

$$|(2+3i)-(1+i)| = |1+2i| = \sqrt{1^2+2^2} = \sqrt{5}.$$

Ex: Describe geometrically the set $\{z : \underbrace{|z-1-i|}_{|z-(1+i)|} = 2\}$.

$$\underbrace{|z-(1+i)|}_{\text{the distance between } z \text{ and } (1+i).}$$



Complex Conjugation;

Def 3; Let $z = x + iy$. The complex conjugate $\bar{z}$ of z is the complex numbers $\boxed{\bar{z} = x - iy.}$

Ex: $\overline{2 + 3i} = 2 - 3i$

Important Identity!

Let $z = x + iy$.

$$z\bar{z} = (x + iy)(x - iy) = \underbrace{\underbrace{(x^2 + y^2)}_{|z|^2} + i\underbrace{(x(-y) + yx)}_{0}}_{\text{Real Number!}}$$

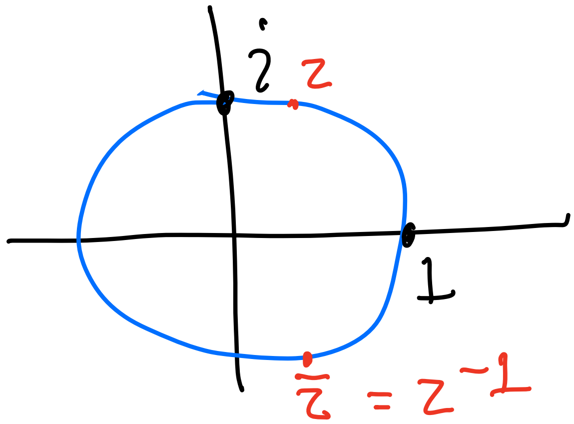
$$\boxed{z\bar{z} = |z|^2}$$

If $z \neq 0$, then

$$z \cdot \frac{\bar{z}}{|z|^2} = 1. \text{ So}$$

$$\boxed{z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}}$$

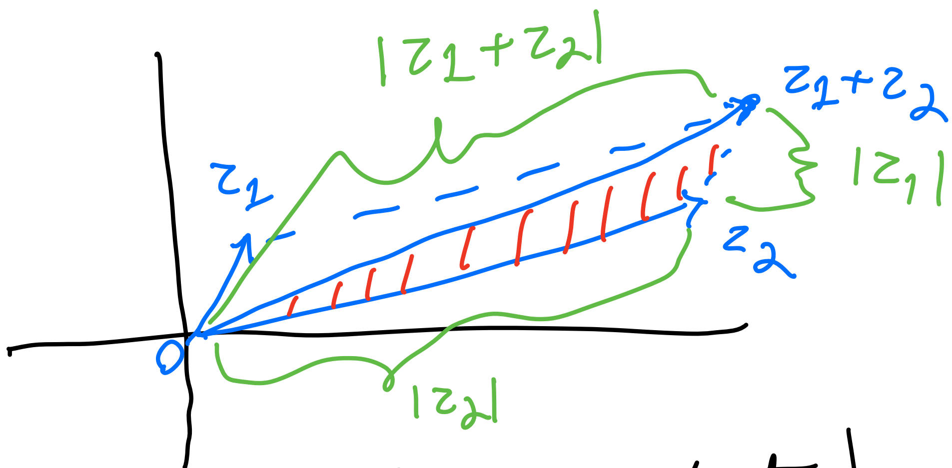
Ex: On the unit circle $\{z : |z| = 1\}$



TRIANGLE INEQUALITY:

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$



Thm: (Triangle Inequality)

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

and equality holds if and only if one of the two cpx numbers z_1, z_2 is a non-zero real multiple of the other.

Cor: $\boxed{|z_1+z_2| \overset{*}{\geq} \big||z_1|-|z_2|\big|}$

Proof: $|z_1| = |(z_1+z_2)-z_2| = |(z_1+z_2)+(-z_2)|$

$$\leq |z_1+z_2| + \underbrace{|-z_2|}_{|z_2|}$$

$|z_1| \leq |z_1+z_2| + |z_2|$. So

$$\boxed{|z_1+z_2| \geq |z_1|-|z_2|}.$$

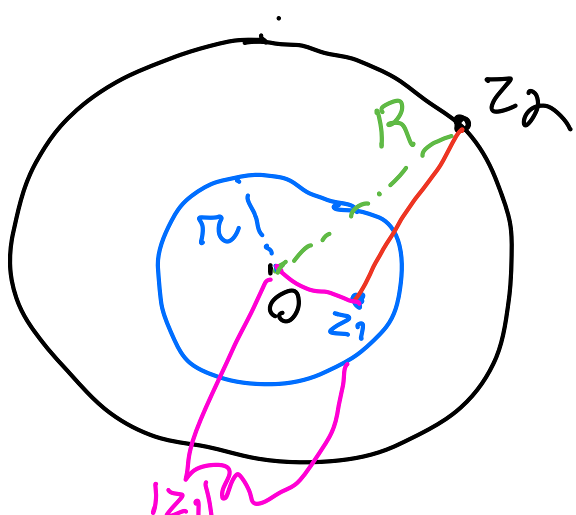
Interchanging the roles of z_1 and z_2 we get

$$\boxed{|z_1+z_2| \geq |z_2|-|z_1|}.$$

So indeed

(*) hold. Q.E.D

Ex: Let z_1 be a complex number on the circle of radius R centered at 0,



Let z_1 be a complex number inside the disk of radius r centered at 0, show that the distance between z_1 and z_2 is larger than $R - r$.

$$|z_2 - z_1| > R - r.$$

$$|z_2 - z_1| \geq \underbrace{|z_2|}_{= R} - \underbrace{|z_1|}_{< r} > R - r.$$