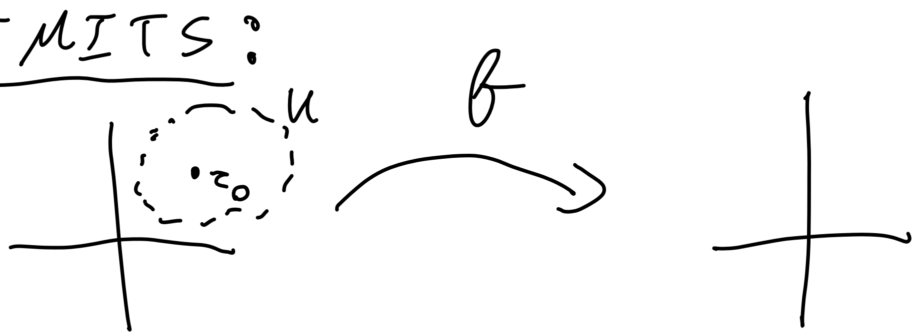
LIMITS:

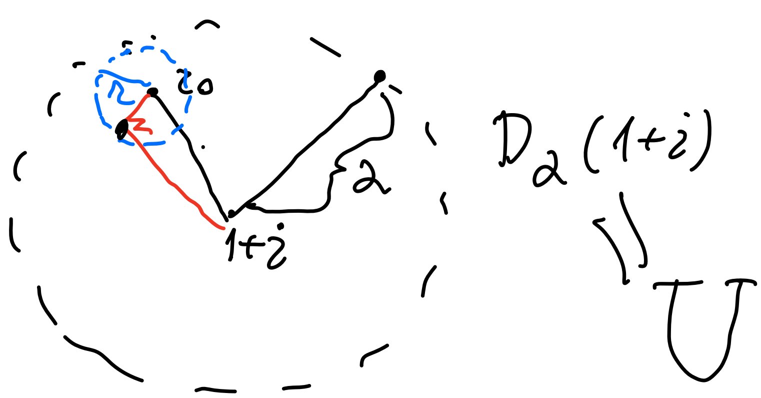


Def 1: A subset U of the complex plane is OPEN, if for every point z_0 in U there is a sufficiently small radius r, such that the open disk of radius r centered at z_0

$$D_r(z_0) = \{ z : |z - z_0| < r \}$$

is contained in U.

Ex: The open disk $D_2(1+i)$ of radius 2 centered at $1+i$ is open.



$|z_0 - (1+i)| < 2$

Let $r = 2 - |z_0 - (1+i)|$.

Then the disk $D_r(z_0)$ is contained in U.

$r = 2 - |z_0 - (1+i)|$

$$\underbrace{|z-(1+i)|}_{=(z-z_0)+(z_0-(1+i))} \le |z-z_0| + |z_0-(1+i)| <$$

$$r + |z_0-(1+i)| = 2 \underbrace{- |z_0-(1+i)| + |z_0-(1+i)|}_{0}$$

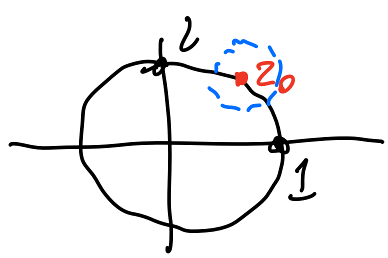
$$= 2.$$

So z is in $D_2(1+i)$.

Note: The closed disk $\overline{D}$

$\{z \,;\, |z| \le 1\}$

is not open.



Let U be an open subset of $\mathbb{C}$. A complex function $f: U \to \mathbb{C}$ defined over U is simply a map from U to $\mathbb{C}$. Explicitly, we can write

$$\boxed{f(x+iy) = u(x,y) + i\,v(x,y)},$$

where $u, v : U \longrightarrow \mathbb{R}$ are real valued functions defined over U.

<u>Ex:</u> Let $U = \mathbb{C}$. Let $f(z) = z^2$. Then

$$f(x+iy) = (x+iy)(x+iy) = \underbrace{(x^2-y^2)}_{u(x,y)} + i\,\underbrace{(2xy)}_{v(x,y)}$$

In "$\mathbb{R}^2$ - notation"

$$f(x,y) = (x^2-y^2,\ 2xy).$$

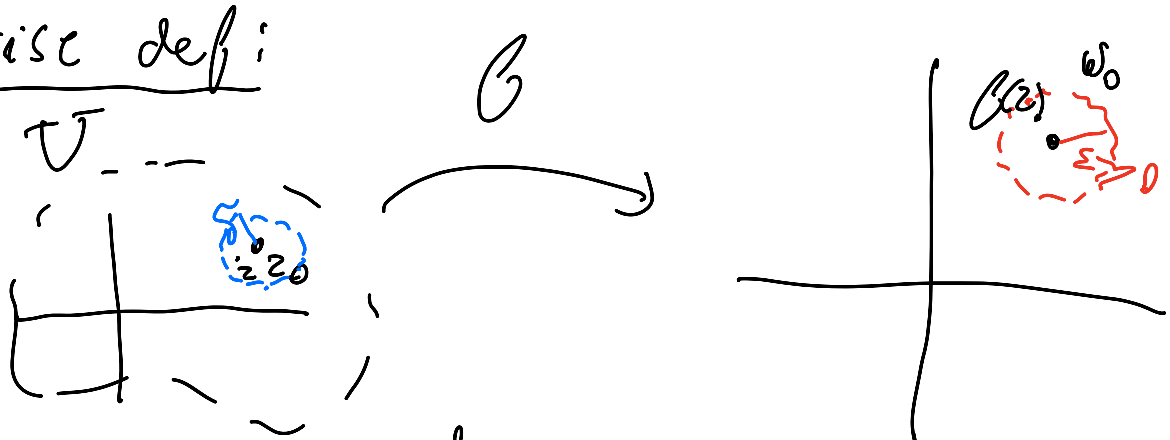
Def 2: Let U be an open subset of $\mathbb{C}$, Let $f: U \to \mathbb{C}$ be a complex valued function. We say that the LIMIT of $f(z)$ at z_0 in U exists and is equal to w_0, and write

$$\lim_{z \to z_0} f(z) = w_0,$$

if

(Intuitively): $f(z)$ can be made arbitrarily close to w_0, if we choose z sufficiently close to z_0, but not equal to z_0.

Precise def:



For every $\varepsilon > 0$ real, there exists a real number $\delta > 0$, such that

$$\boxed{f\left(D_\delta(z_0) \text{ minus } z_0\right) \subset D_\varepsilon(w_0)}$$

$$\left(\Leftrightarrow \boxed{\text{if } 0<|z-z_0|<\delta, \text{ then } |f(z)-w_0|<\varepsilon}\right).$$

<u>Def 3</u>; Let f be as in Def 2.
We say that f is CONTINUOUS at z_0 in U, if $\lim_{z\to z_0} f(z) = f(z_0)$.

<u>Remark</u>; (Fact)
If f is as in Def 2,
and $f(x+iy) = u(x,y) + i\,v(x,y)$
and $z_0 = x_0 + i y_0$, $w_0 = u_0 + i v_0$,
then $\lim_{z\to z_0} f(z) = w$
if and only if
$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ AND $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

Ex: The function $f(z) = z^2$ is continuous at every point z_0 of $\mathbb{C}$;

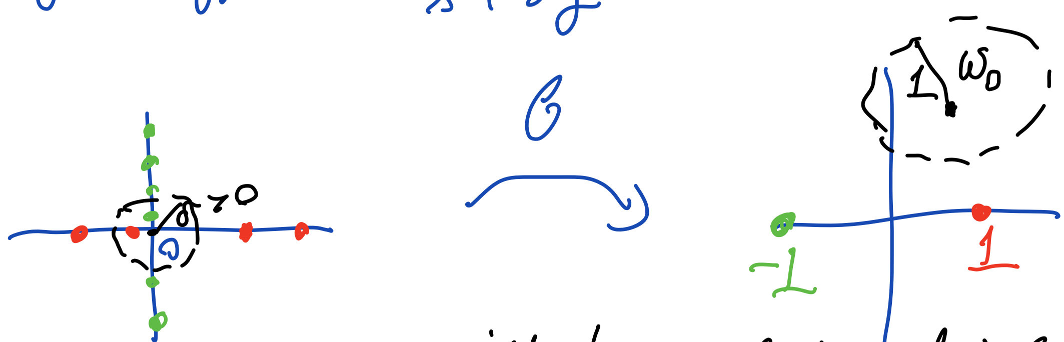
Calc 3 method of proof:

$$f(x+iy) = \underbrace{(x^2-y^2)}_{u(x,y)} + i\,\underbrace{2xy}_{v(x,y)}$$

and u, v are just poly in x,y, so continuous everywhere.

Ex: Let $f(z) = \bar{z}/z$. Let $U = \mathbb{C}$, Then $\lim_{z \to 0} f(z)$ does not exist.

$$f(x+iy) = \frac{x-iy}{x+iy}$$



Let $w_0 \in \mathbb{C}$, we will show that $\lim_{z \to z_0} f(z)$ is not w_0

Let $\varepsilon = 1$ in the def of LIMIT.

Observation 1: At least one of the

numbers 1 or -1 is not in $D_1(w_0)$

Observation 2: Every open disk centered at 0 of radius $\delta > 0$, no matter how small δ is, intersects both the real axis and the imaginary axis. So 1 and -1 are both values of f on some points in $D_\delta(0)$. So $f(D_\delta(0) \text{ minus } 0)$ is not contained in $D_1(w_0)$.

Q.E.D.

Remark: The usual limit laws from Calc 1 hold also for complex valued functions of a complex variable. The proofs are the same.

Example: (The Quotient Limit Law)

Let f, g be complex valued functions defined on some open subset U of $\mathbb{C}$ and z_0 a point of U.

Assume that $\lim_{z \to z_0} f(z)$ and $\lim_{z \to z_0} g(z)$ exist and the latter is non-zero. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)}.$$

Derivatives:

Def 4: Let f be a complex valued function defined on some open subset U of $\mathbb{C}$ and z_0 a point of U.

The DERIVATIVE of f at z_0 is

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided the limit exists.

In this case we say that f is DIFFERENTIABLE at z_0.

Remark: If f is differentiable at z_0, then f is continuous at z_0

Proof: It suffices to show that

$$\boxed{\lim_{z \to z_0} f(z) - f(z_0) = 0.}$$

Indeed,

$$\lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} (z - z_0) \cdot \frac{f(z) - f(z_0)}{z - z_0} \overset{\uparrow}{=}$$

$$\boxed{\text{Product Limit Law}}$$

$$\underbrace{\lim_{z \to z_0} (z - z_0)}_{0} \cdot \underbrace{\lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right)}_{\text{exists, by assumption and equal } f'(z_0)} =$$

$$= 0 \cdot f'(z_0) = 0.$$

Q.E.D

Ex: $f(z) = z$ is differentiable and $f'(z_0) = 1$.

Ex: Let $g(z) = \bar{z}$. Then g is not differentiable at 0.

Reason: $\lim\limits_{z \to 0} \frac{g(z) - g(0)}{z - 0} =$

$$= \lim_{z \to 0} \frac{\bar{z} - 0}{z - 0} = \lim_{z \to 0} \frac{\bar{z}}{z}$$ does not exist, by the previous example.

Note: $g(z) = \bar{z}$ is not differentiable anywhere. This will follow from the Cauchy-Riemann Eq next lecture.

Ex: $f(z) = z^n$, n a positive integer, is differentiable everywhere.

This is a consequence of the fact that the usual derivative rules from Calc I are valid. (sum, product, quotient, etc).

Example: (Product Rule)

Theorem: If f, g are differentiable at z_0, then so is $f \cdot g$, and

$$(fg)'(z_0) = \boxed{f'(z_0)g(z_0) + f(z_0)g'(z_0)}.$$

Proof:

set $\Delta z := z - z_0$

$$\lim_{z \to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} \overset{\downarrow}{=}$$

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)g(z_0+\Delta z) - f(z_0)g(z_0)}{\Delta z} =$$

$$\lim_{\Delta z \to 0} \frac{\left[f(z_0+\Delta z)g(z_0+\Delta z) - f(z_0+\Delta z)g(z_0)\right] + \left[f(z_0+\Delta z)g(z_0) - f(z_0)g(z_0)\right]}{\Delta z}$$

Sum limit rule

$$= \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)\left[g(z_0+\Delta z) - g(z_0)\right]}{\Delta z} + \lim_{\Delta z \to 0} \frac{g(z_0)\left[f(z_0+\Delta z) - f(z_0)\right]}{\Delta z}$$

product rule

$$= \underbrace{\lim_{\Delta z \to 0} f(z_0+\Delta z)}_{f(z_0)} \underbrace{\lim_{\Delta z \to 0} \frac{g(z_0+\Delta z) - g(z_0)}{\Delta z}}_{g'(z_0)} + g(z_0) \underbrace{\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}}_{f'(z_0)}$$

$$= f(z_0)\, g'(z_0) + g(z_0)\, f'(z_0).$$

Q.E.D

<u>Cor:</u> $f(z) = z^2$ is differentiable everywhere.

<u>Proof</u>: We know that $f(z) = z$ is. So, by the product rule, so is

$f(z) \cdot g(z)$.

Ex: Every poly

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$$

$a_0, \dots, a_n$ constant cplex numbers

is differentiable everywhere

Ex: Every rational function

$R(z) = \dfrac{f(z)}{g(z)}$ where f, g are poly in z, is differentiable at any point z_0 which is not a root of g.